Credit Risk

Lecture 1 – Introduction, reduced-form models and CDS

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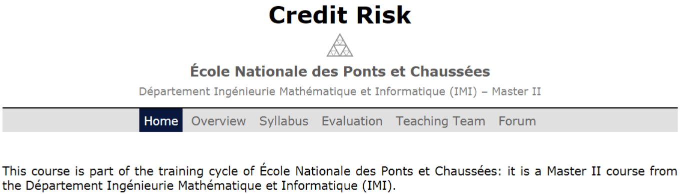
Class structure and assignments

Class structure and assignments

Class website

All the information is on the **website**:

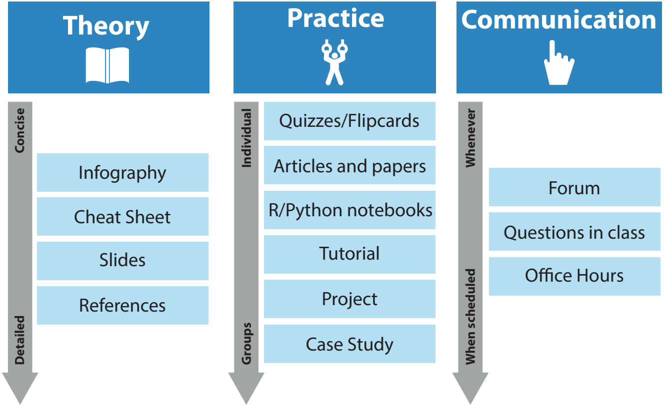
defaultrisk.free.fr



Teaching team, syllabus, grading, etc.

Class structure and assignments

Pedagogical tools (I/II)



Class structure and assignments

Pedagogical tools (II/II)

Buttons	Blocks	Other styles
Quiz	**Definition - Math** I use this block for math definitions.	This is a proof.
R Markdown	**Definition - Eco or finance** I use this one for economics or finance definition.	This is the solution of an exercice, or details of an explanation.
Tutorial	**Be careful!** I use this one to draw your attention.	I am citing: [Harrison and Kreps, 1979].
Newspapers	**Example** I use this one when giving concrete examples.	I will **emphasize** important words.
Be Careful!		
Theorem		
Definition		

Objectives of the lecture

Teaching objectives

At the end of this lecture, you will:

- Understand why credit risk is at the **basis of our economies**;
- Have a clear view on the credit risk modeling **challenges and outcomes**;
- Know the **basic concepts of credit risk**, that is, how to price a bond, what a spread is and how to extract it from the price of a bond, what are the Exposure At Default (EAD), Loss Given Default (LGD) and Probability of Default (PD);
- Know what **reduced-form models** are and how to calibrate them;
- Know what **Credit Default Swaps** are and how to price them.

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Credit risk and economics

What is credit risk?

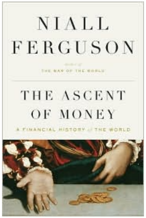
credo: I believe (latin)
resecare: To break (latin)

Credit risk – Definition

Credit risk is the risk of **default** on a debt, that may arise from a borrower failing to make required payments. [BCBS, 2000]

[Fergusson, 2008] is an interesting reference to tackle the subject from an historical perspective.

▸ YouTube



Credit risk and economics

Why is there credit risk?

There is a **discrepancy of financial needs** among economic agents.
Some agents need money to fulfill their projects (firms, states, people, etc.) and other do not need an immediate access to their wealth.

↓

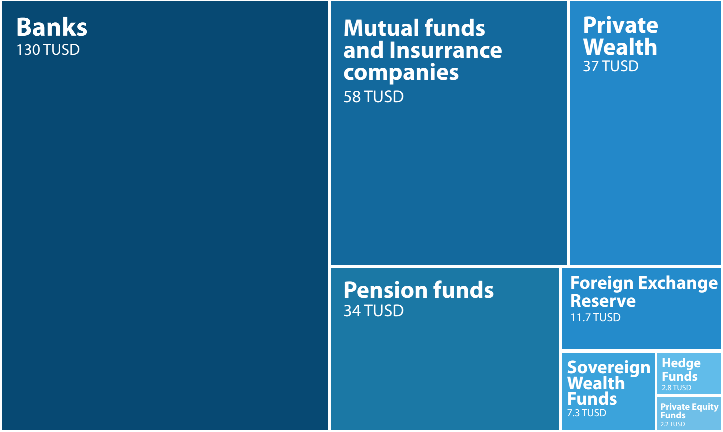
To fill this gap, lenders lend to borrowers, based on the **belief** that they will retrieve their money and get adequate reward.

↓

This belief – this trust – is at **the origin of credit risk**.

Credit risk and economics

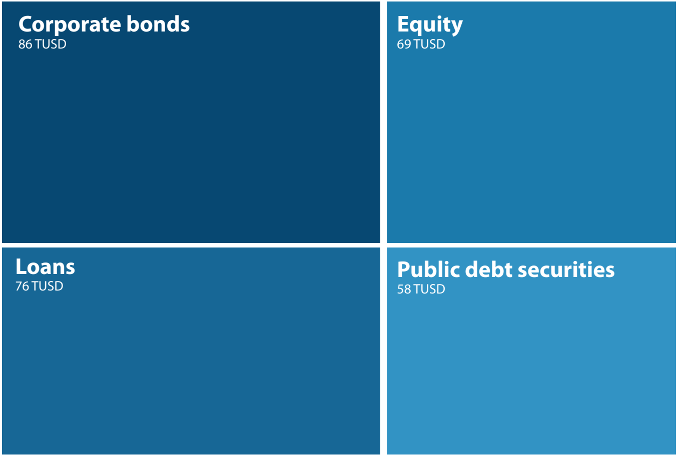
Who finances the economy?



Source: Aspects of Global Asset Allocation, IMF. and own cross-checkings.

Credit risk and economics

Who borrows?



Source: The Random Walk, Mapping the world financial markets, 2014, DB research.

Credit risk and economics

In what banks differ from other lenders?

They have an **expertise** and a defined economic purpose as financial intermediaries.

- They have an expertise in **maturity transformation** (ALM[1] department);
- They have much more **information** on the economy and on their counterparties than any other agent;
- They know how to dissociate risks and underlying assets thanks to **derivative products**;
- They can deal with credit risk on a **macro level** (portfolio approach, dynamic management of assets, macro hedging strategy);
- They **create money** when granting credits.

See several references at the end of the slides.

[1] Asset Liability Management.

Credit risk and economics

Regulatory requirements for banks

Banks are at the center of our economies

- Finance the economy;
- Provide way to transfer risks;
- At the basis of money creation.

Banks face numerous and complex risks

- Credit Risk (~80 %*);
- Market Risk (~10 %*);
- Liquidity Risk;
- Operational Risk (~10 %*).

* Computed as a percentage of Risk Weighted Assets (RWA, see Lecture 5).

It is thus a highly regulated sector

Banks must set capital apart to face an unlikely rise of these risks.

↓

Incentives to reduce balance sheet size

↓

Moving towards an *originate to distribute* model ? (vs *originate to hold*)

Conclusion
Credit risk and economics

- Credit risk is the risk that a **borrower fails to make required payments or has a higher risk not to repay**;
- There is **no financing of the economy without credit risk**;
- **Banks finance** a big chunk of the economy and are thus systematic actors exposed to credit risk.

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The outcomes we will face in this class

Very different outcomes in comparison with market risk

	Market Risk	Credit Risk
Amount of data	A lot	Few
Liquidity of the assets	Liquid	Not liquid
Shape of the loss function	Symmetric	Asymmetric
Correlations	High	Low
Risk Management	Hedging	Diversifying
Backtesting	Possible	Impossible

The outcomes we will face in this class

The outcomes we will face in this class

Estimation of the prob. of default	**Point** in time	vs	**Through** the Cycle
Time horizon	**One** Year	vs	**Several** Years
Accountancy	**In** balance-sheet	vs	**Off** balance-sheet
Number of counterparties	**Single-name** models	vs	**Portfolio** models
Assets	**With** assets	vs	**Without** assets
What to predict?	**Two-state** model	vs	**Continuous** model
Purpose	**Regulatory** purpose	vs	**Internal** purpose
Probability	**Risk Neutral** Probability	vs	**Real World** Probability
Data	**Lack** of data	vs	**Lot** of data

Let us take a closer look at the two latter.

Real World and Risk Neutral probabilities

Real World and Risk Neutral probabilities – Definitions

Let S_t be the variable equals to s, if the future state of the economy, t.

Real World Probability, $\mathbb{P}$

Probability that an event, s, occurs.

Risk Neutral Probability, $\mathbb{Q}$

Probability measure which weights the future state of the economy, s, according to **the price to be risk neutral to that specific state**, proportionally to the price to be risk neutral to all the future states of the economy.

This formalization was made by [Harrison and Kreps, 1979].

Real World and Risk Neutral probabilities

A simplified example to understand Risk Neutral Probability

A simplified example – Real World and Risk neutral probabilities

Definition

Let us say that a future state of the economy (in 3 years) will occur with a (Real World) probability of 10 %. Let us suppose that:

- to be sure (i.e. to be risk neutral) to have a cash flow of 1 EUR in 3 years (that is the price of a risk-free zero-coupon bond) costs 0.8 EUR;
- to be sure (i.e. to be risk neutral) to have a cash flow of 1 EUR, only if, the specific state s of the economy occurs in 3 years, costs 0.1 EUR.

We have that: $\mathbb{Q}(S_t = s) = \frac{0.1}{0.8} = 12.5$ % even if $\mathbb{P}(S_t = s) = 10$ %.

In that case, the cost of protecting oneself against s is higher than suggested by the Real World probability.

Real World and Risk Neutral probabilities

Risk Neutral Default rates vs Real World Default rates

Rating (in %)	Real World Default rate	Risk Neutral Default rate
AAA	0.03	0.60
AA	0.06	0.73
A	0.18	1.15
BBB	0.44	2.13
BB	2.23	4.67
B	6.09	8.02
CCC	13.52	18.39

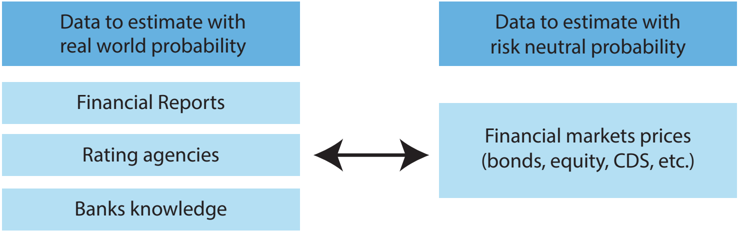
Risk Neutral Default rates are higher than Real World Default rates.

That could be because of:

- The **lack of liquidity** on the debt market;
- The **lack of information of investors** on the market;
- The **risk aversion of investors** on the market, etc.

Credit risk modeling and the challenge of data

Where to find data for credit risk modeling?



Conclusion

Main credit risk modeling outcomes and challenges

- Contrary to market risk, credit risk faces the following **challenges**: there is few data, the market is illiquid, the loss functions are asymmetric, correlations are low and backtesting is hardly possible;
- Additionally, credit risk can be **envisioned in many different ways**: on several time spans, with a real-world or risk neutral approach, in a continuous or binary perspective, etc. making this risk particularly technical.

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Pricing of bonds – Continuous version

The spread of a bond

Pricing a bond – Continuous version

Proof

Let us denote the continuous and constant coupon rate by c, the risky rate of the firm A by r^A, the maturity of the bond T, and assume the nominal, N, is equal to 1, **the price of a bond** is:

$$\bar{B}^A(0, c, T) = 1 + (c - r^A)\frac{1 - e^{-r^A T}}{r^A}$$

> The formula is a simple result consequent to the no-arbitrage assumption and the integral resolution of: $\bar{B}^A(0, c, T) = \int_0^T ce^{-r^A t}dt + Ne^{-r^A T}$

The spread of a bond – Continuous version

For a given risk-free rate, r, the spread of a bond of price $\bar{B}^A(0, T)$, is the value s^A so that:

$$\bar{B}^A(0, c, T) = 1 + (c - (r + \boldsymbol{s^A}))\frac{1 - e^{-(r+\boldsymbol{s^A})T}}{(r + \boldsymbol{s^A})}$$

Pricing of bonds

Pricing and implied survival probability

The implied survival probability – Computed with prices

Proof

Let τ be the time of default of firm A. Let $\bar{B}^A(t,0,T) = \bar{B}^A(t,T)$ be the price of a zero-coupon risky bond of firm A, at t, of maturity T, and nominal $N = 1$. Let $B(t,0,T) = B(t,T)$ be the price of a zero-coupon risk-free bond, at t, of maturity T, and nominal N.

The implied survival probability of firm A, in T, from t, is:

$$\mathbb{Q}(\tau > T \mid \tau > t) = \frac{\bar{B}^A(t,T)}{B(t,T)}$$

It is a consequence of the no-arbitrage assumption.

The implied survival probability – Computed with constant continuous spreads

Let s^A be the spread of Firm A. The **implied survival probability** can be written:

$$\mathbb{Q}(\tau > T \mid \tau > t) = e^{-s^A(T-t)}$$

$$\mathbb{Q}(T > \tau \mid \tau > t) = \frac{Ne^{-r(T-t)}}{Ne^{-(r+s^A)(T-t)}} = e^{-s^A(T-t)}$$

Tutorial | Notebook

Three points of attention – Discrete version, recovery and risk-free rate

Continuous vs Discrete (I/IV)

Pricing a bond – Discrete version

Let N be the nominal of a bond from firm A, $t_1, ..., t_n = T$ the dates when the coupons C are paid, $r_1^A, ..., r_n^A$ the respective risky rates and, T, its maturity. The **price of the bond** $\bar{B}^A(0, c, T)$, in 0, is:

$$\bar{B}^A(0, c, T) = \sum_{i=1}^{n} \frac{C}{(1 + r_i^A)^{t_i}} + \frac{N}{(1 + r_T^A)^T}$$

The spread of a bond – Discrete version

Let $\bar{B}^A(0, C, T)$ be the price of a bond and $r_1, ..., r_n$ the risk-free rates in $t_1, ..., t_n$. **The spread** of A is s^A so that:

$$\bar{B}^A(0, C, T) = \sum_{i=1}^{n} \frac{C}{(1 + r_i + \mathbf{s^A})^{t_i}} + \frac{N}{(1 + r_n + \mathbf{s^A})^T}$$

Three points of attention – Discrete version, recovery and risk-free rate

Continuous vs Discrete (II/IV)

Bootstrap technique to compute implied survival probability – Discrete version

Suppose firm A has n bonds, $B_1^A, ..., B_n^A$, paying coupons, c, in $t_1,...,t_n$ and of maturity, respectively $t_1,...,t_n$. **Bootstrap** works the following way:

- With B_1^A, it is possible to compute r_1^A;
- With B_2^A, subtracting its cash-flow in t_1 discounted by $1 + r_1^A$, it is possible to compute r_2^A;
- etc.

That way, one can compute iteratively $r_1^A, r_2^A, ..., r_n^A$ and thus deduce $s_1^A, s_2^A, ..., s_n^A$ using the risk-free rates.

Three points of attention – Discrete version, recovery and risk-free rate

Continuous vs Discrete (III/IV)

Bond clean and dirty prices

Discrete payments implies discontinuity in bond valuation each time a coupon is paid : these discountinuous prices are called **dirty prices**. On the markets, the prices do not suffer such a problem as the so-called **clean prices** are quoted. The formula that links both is:

$$\text{Dirty price} = \text{Clean price} + \text{Accrued interests}$$

where $\text{Accrued interests} = \frac{\#\text{ days since last coupon}}{\#\text{ days between coupons}} \times \text{Coupon rate} \times \text{Nominal}$

Three points of attention – Discrete version, recovery and risk-free rate

Continuous vs Discrete (IV/IV)

Adobe System Inc. bond valuation

Adobe Systems Inc. (NASDAQ: ADBE) has 600 MUSD worth of bond payable outstanding. The 1 000 USD par, 3.25 % semi-annual coupon bonds are due to mature on 1st February 2015. The coupon dates are 1st February and 1st August. They follow 30/360 day count convention and next coupon is due on 1st August 2013. Yvonne Barnet bought 1 000 such bonds from Charles Schwab on 20th July 2013. The market requires buyer to compensate seller for the accrued interest. How much Yvonne must pay Charles? Yvonne must pay the dirty price, but she only knows the clean price: 1036.10 USD.

- days between the transaction date and next coupon date = 11 = 10 days of July plus 1 day of August;
- days in the coupon period = 180 (since 30/360 day count convention is used).

Thus, the dirty price is: $1036.10 + \frac{1000 \times 3.25\%}{2} \times \frac{169}{180} = 1051.36$ USD.

Three points of attention – Discrete version, recovery and risk-free rate

The importance of the recovery rate

The importance of the recovery rate

To simplify math formulas, the recovery rate – the extent to which principal and accrued interests on defaulted debt can be recovered, expressed as a percentage of face value – is often forgotten (or equivalently supposed equal to one). Nonetheless, in practice, **the recovery rate must be taken into account** when extracting the probability of default from a market price.

▸ Newspapers

The implied probability of default taking into account recovery

▸ Proof

Let R be the recovery rate, the **implied probability of default taking into account recovery**, from now, to T, is:

$$\mathrm{PD}_{0,T} = \frac{1 - \frac{\bar{B}(0,T)}{B(0,T)}}{1 - R}$$

▸ Be Careful!

This is a consequence of the no-arbitrage hypothesis.

▸ Tutorial

Three points of attention – Discrete version, recovery and risk-free rate

What is the risk-free rate?

What is the risk-free rate?

The rates used as **risk-free rates** evolved during the last decades:

- from the government rates, first;
- to the LIBOR rates, then;
- to the Overnight Indexed Swap (OIS) rate, now.

ICE Swap Rate

The general framework of credit risk modeling: PD, EAD and LGD

Definitions of PD and EAD

Probability of Default – PD

It is the **one-year default probability** of a counterparty.

Exposure At Default – EAD

Exposure At Default is the **loss** a bank would suffer if its counterparty defaults, and there would be no guarantee.

This value can either be **known** (loans for example) or **unknown** (lines of credit for instance).

The general framework of credit risk modeling: PD, EAD and LGD

Focus on the Loss Given Default (I/II)

Loss Given Default – LGD

Let R be the **recovery rate**, that is the percentage of the exposure recovered by the bank after the default, we have:

$$\text{LGD} = 1 - R$$

Loss Given Default and R vary depending of the underlying credit contract

Contracts	**R**
Bank loans	80.3 %
Senior secured bonds	63.5 %
Senior unsecured bonds	49.2 %
Senior subordinated bonds	29.4 %
Subordinated bonds	29.5 %
Junior subordinated	18.4 %

Source: Moody's statistics.

▸ Moody's

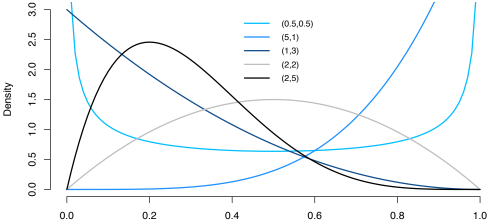
The general framework of credit risk modeling: PD, EAD and LGD

Focus on the Loss Given Default (II/II)

Modeling LGD with a beta distribution using a Maximum Likelihood estimator

Let $\alpha > 0$, $\beta > 0$. The density of a beta distribution is:

$$f(x;\alpha,\beta) = \begin{cases} \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,\dot{u}} & \text{for } x \in [0,1] \\ 0 & \text{else} \end{cases}$$



From data, it appears that the shape of LGD distributions is usually a **U-shaped curve**.

The general framework of credit risk modeling: PD, EAD and LGD

Independence of PD, EAD and LGD

The Expected Loss – EL

We define the Expected Loss as (on time horizon T):

$$\begin{aligned}\text{EL} &= \mathbb{E}(\text{EAD} \times \mathbb{1}_{\{\tau<T\}} \times \text{LGD}) \\ &\underbrace{=}_{\text{assump. } \perp} \mathbb{E}(\text{EAD}) \times \underbrace{\mathbb{E}(\mathbb{1}_{\{\tau<T\}})}_{\text{PD}} \times \mathbb{E}(\text{LGD})\end{aligned}$$

Be Careful!

The independence of EAD, PD and LGD

Formula

There is no reason why one should assume **independence** between EAD, PD and LGD. Actually, some phenomena and papers proved the contrary:

- The phenomenon of **gambling for resurrection**;
- The study by [Frye, 2013] postulates that **LGD is a function of the probability of default**.

Quiz

Conclusion
The basics of credit risk

- Pricing of bonds, in a continuous or discrete framework, is based on the **no-arbitrage assumption**;
- **Coupons and recovery** are complexities that need to be taken into account when implying probabilities of default from bond prices through respectively accrued interests cleaning and bootstrapping;
- There are **three key components** of any credit expected losses estimator, the Exposure At Default, the Loss Given Default and the Probability of Default – their independence, while often assumed in models, is not always tenable;
- The **Probability of Default (PD)** is the major focus of this course.

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Reduced-form models

Definition

Reduced-form models

Reduced-form models consist in modeling the **conditional law of the random time of default**.

Conditional default probability

In a reduced-form model, **conditional default probability** is defined as:

$$\mathbb{Q}(\tau < t + \mathrm{dt} \mid \tau > t) = \lambda \mathrm{d}t$$

λ is the **default intensity** and corresponds to the **instantaneous forward default rate**. This variable is **exogenous** to the problem.

Survival function

▸ Proof

In a reduced-form model, with λ constant, the **survival function** is defined as:

$$S(t) = \mathbb{Q}(\tau > t) = \exp(-\lambda t)$$

τ, the the time of default, follows an Exponential distribution of parameter λ.

Reduced-form models

What is default intensity?

Is the default intensity, λ, constant or stochastic?

It depends:

- **Constant**: Time homogeneous Poisson Process;
- **Deterministic**: Time deterministic inhomogeneous Poisson Process;
- **Stochastic**: Time-varying and stochastic Poisson Process as the Cox, Ingersoll, Ross (CIR) model.

Reduced-form models

Calibration of default intensity models

The implied survival probability

Let $B(0,t)$ be a **zero-coupon risk-free bond** and $\bar{B}(0,t)$ be a risky zero-coupon bond. We have:

$$\mathbb{Q}(\tau > t) = \frac{\bar{B}(0,t)}{B(0,t)}$$

This is a result based on the no-arbitrage assumption.

The implied survival probability – Application for calibration of intensity models

We deduce from the above formula the expression of the **default intensity**, λ:

$$\lambda = -\frac{\log\left(\frac{\bar{B}(0,t)}{B(0,t)}\right)}{t}$$

Be Careful!

We have seen that, $\mathbb{Q}(\tau > t) = e^{-\lambda t}$

Conclusion

Reduced-form models

- Reduced-form models are models based on an **exogenous variable**, called here, the default intensity;
- The **default intensity is often assumed constant** but can also be non-constant or even stochastic.

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Single-name credit derivatives and Credit Default Swap (CDS)

What are CDS?

Credit Default Swap

Credit Default Swaps (CDS) are financial agreements that allow the **transfer of the credit risk** of a loan to another counterparty.

- Bank A holds a loan on corporate C on its balance-sheet and buys protection on the credit risk related to counterparty C;
- Bank B sells protection on corporate C and receives the payment of a premium.

Are CDS insurance contracts?

CDS are **neither insurance contracts, nor guarantees**.

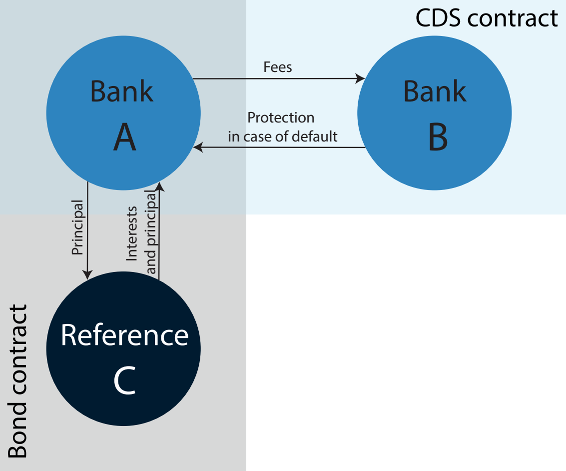
CDS and JP Morgan

JP Morgan pioneered the CDS, more precisely, Blythe Masters is considered the inventor of this security.

▸ Wikipedia

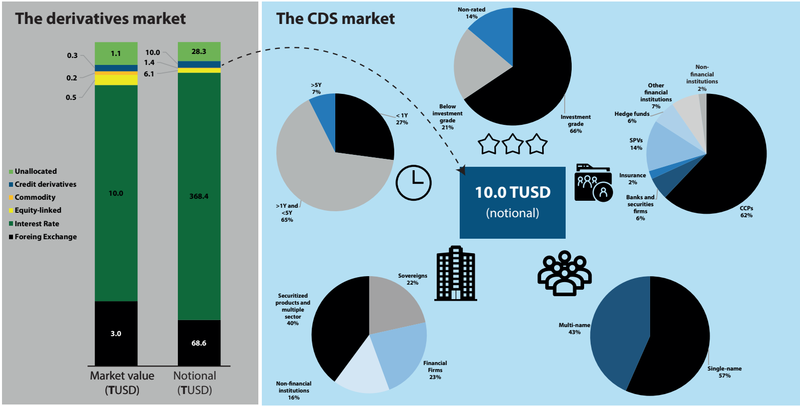
Single-name credit derivatives and Credit Default Swap (CDS)

CDS cash flows



Statistics on the CDS market

The CDS market



Legal aspect of CDS

The International Swaps and Derivatives Association (ISDA)

ISDA defines **standards** for credit derivatives transactions:

- **1999**: definitions;
- **2003**: supplements;
- **2009**: big-bang protocol.

▸ ISDA

Specifications for a CDS

Examples of important specifications for a CDS

- What is a CDS **credit event**?
  - bankruptcy;
  - failure to pay (coupon or nominal);
  - restructuring of a bond.
- **Settlement**:
  - Physical settlement: the buyer of the CDS gives the defaulted bonds to the seller, and receives the nominal (N);
  - Cash settlement: the buyer receives $N(1 - R)$, and keeps the defaulted bonds.
- **Normal vs Digital CDS**:
  - Digital: the payoff is N;
  - Normal: the payoff is $(1 - R)N$.

Various definitions of default depending on seniority

Definitions of default for financial institutions in Europe after the crisis

The new banking regulations that followed the 2008 to 2011 economic crisis had consequences on the financial ratios that banks should respect. The **ISDA adapted its definition of default** to embrace the existence of these new ratios.

UniCredit Note

Several CDS for one company

One company can have different Credit Default Swaps adapted to several maturities and debt seniorities: that is the reason why you will find several CDS references for the same company.

The Ice

Single-name credit derivatives and Credit Default Swap (CDS)

Use of CDS

What CDS can be used for?

CDS are mostly used for **risk management and investment strategies**.

- Risk mitigation;
- Management of credit lines;
- Bank's capital management;
- Balance sheet management;
- Leverage effect;
- Access to the market.

Who uses CDS?

Very few corporate firms and retail customers use these securities, CDS are mostly used by **financial institutions**:

- Banks;
- Corporate firms;
- Insurers and reinsurers;
- Asset managers;
- Hedge funds.

Reduced-form models applied to CDS pricing

Fixed leg of a CDS

Value of the fixed leg of a CDS

The value of the **fixed leg** of a CDS is:

$$\text{Fixed}(0,T) = s(0,T)\frac{1-e^{-(r+\lambda)T}}{r+\lambda}$$

Fixed leg pays the reference spread at inception of the CDS up to the minimum of the default date (τ) and maturity T.
For a continuous paid spread:

$$\begin{aligned}\text{Fixed}(0,T) &= \mathbb{E}^{\mathbb{Q}}\left(s(0,T)\int_0^{\tau\wedge T} e^{-rt}\mathrm{d}t\right)\\ &= s(0,T)\int_0^{T} e^{-(r+\lambda)t}\mathrm{d}t\\ &= s(0,T)\frac{1-e^{-(r+\lambda)T}}{r+\lambda}\end{aligned}$$

Reduced-form models applied to CDS pricing

Floating leg of a CDS

Value of the floating leg of a CDS

The value of the **floating leg** of a CDS is:

$$\text{Floating}(0, T) = (1 - R)\frac{\lambda}{\lambda + r}\left(1 - e^{-(\lambda+r)T}\right)$$

The floating leg is paid when a default occurs: the protection seller pays the difference between the notional and the recovery.

$$\begin{aligned}\text{Floating}(0, T) &= \mathbb{E}^{\mathbb{Q}}\left((1 - R)e^{-r\tau}\mathbb{1}_{\{\tau < T\}}\right) \\ &= (1 - R)\frac{\lambda}{\lambda + r}\left(1 - e^{-(\lambda+r)T}\right)\end{aligned}$$

Reduced-form models applied to CDS pricing

Spread of a CDS, its Mark-to-Market value and time sensitivity

The spread of a CDS

▸ Proof

The **spread of a CDS** is:

$$s = \lambda(1 - R)$$

> At inception, the Net Present Value (NPV) for the protection seller is:
>
> $$\text{MtM}(0, T) = \text{Fixed}(0, T) - \text{Floating}(0, T)$$
>
> The fair spread sets the initial MtM at 0. We thus have $s = \lambda(1 - R)$.

MtM through the time and sensitivity of the CDS to the time

The sensitivity of the MtM is the **risky duration**, DV:

$$\text{DV}(t, T, \lambda) = \frac{1 - e^{-(r+\lambda)(T-t)}}{r + \lambda}$$

▸ Tutorial

Reduced-form models applied to CDS pricing

CDS sensitivity

Present Value of a CDS through the time

▸ Proof

Let s_0, be the spread in $t = 0$, and s, the spread today, in t.
The **Present Value of the protection seller** is:

$$PV(s_0, s_t) = \mathrm{DV}(0, t, \lambda)(s_0 - s_t)$$

Reduced-form models applied to CDS pricing

CDS – Discrete vs continuous pricing

CDS – Discrete vs continuous pricing

Once again, these formulas suppose continuous interest rates payments and spread payments. These **formulas are nice proxies** to assess CDS spreads, but for precise pricing, **one must take into account each flow separately**.

▸ Tutorial

Reduced-form models applied to CDS pricing

CDS spread vs Bond spread: the CDS basis

CDS spread and Bond spread

Generally, CDS spreads are larger than Bond spreads.

Several reasons explain this phenomenon:

- definition of credit event is different;
- the protection buyer has no impact on the bond issuer through covenants.

On the other hand, funds and insurers sell massively protection, making CDS spread tighten.

For more information on the subject, you can take a look at [Choudhry, 2006].

Default swaptions and other swaps

An introduction to default swaptions

Knock-out swaptions

Knock-out swaptions:

- Allow to sell or buy protection at maturity;
- Payoff is null if default occurs before maturity.

The pricing is done with models similar to Black-Scholes model.

Default swaptions and other swaps

Other swaps

Other swaps

- **Constant Maturity Credit Default Swaps (CMCDS)**: is a usual CDS except that the *fixed* leg of the contract is computed each semester with the new data. Its *pay-off* is often capped.
- **Loan-CDS (LCDS)**: they are linked to a specific loan, not on a specific name; thus, would the loan be reimbursed in advance, the CDS would be cancelled.
- **Forward CDS**: they are CDS that offer protection from a future date, T to a future data $T + M$.
- **Cancellable CDS**: they are CDS that are cancellable at a or several future dates.
- **Total Return Swap (TRS)**: they are financial contracts that transfer both the credit and the market risk of an asset against either a variable or a fixed rate (they are in fact like funded Interest Rate Swaps – IRS).

Tutorial

Conclusion
Reduced-form models

- CDS are contacts that **protect against default**;
- Their valuations reconcile the **no-arbitrage assumption and the reduced-form models**;
- CDS spreads **can differ from bond spreads**.

Quiz

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Proofs of the lecture

Proofs of the lecture

- Bond price, continuous case
- Implied default probability
- Implied default probability with recovery rate strictly superior to 0
- Survival function in a reduced-form model
- CDS pricing
- Risky duration interpretation

Proof – Bond price, continuous case

$$
\begin{aligned}
\bar{B}^A(0,c,T) &= \int_0^T ce^{-r^A t}\mathrm{d}t + e^{-r^A T} \\
&= \frac{-c}{r^A}\left[e^{-r^A t}\right]_0^T + e^{-r^A T} \\
&= \frac{-c}{r^A}(e^{-r^A T} - 1) + e^{-r^A T} \\
&= \frac{c}{r^A} - \frac{ce^{-r^A T}}{r^A} + e^{-r^A T} \\
&= \frac{c - r^A}{r^A} + 1 - \frac{ce^{-r^A T} - r^A e^{-r^A T}}{r^A} \\
&= \frac{c - r^A}{r^A}(1 - e^{-r^A T}) + 1
\end{aligned}
$$

▸ Theorem

Proof – Implied default probability

Let us assume we have two portfolios, P1 and P2. We take a look at the investment strategy that consists in buying P1 and short-selling P2.Let $PD_{t,T}$ be the probability of default of the considered risky bond between t and T. We have following pay-offs in both Default (D) and Non-Default (ND) scenarios on the risky bond.

Portfolio	Scenarios	In $t=0$	Maturity (T)	Probability of the scena
Portfolio 1 (P1)	Scenario D	$+1$	$e^{r(T-t)}$	$PD_{t,T}$
	Scenario ND	$+1$	$e^{r(T-t)}$	$1-PD_{t,T}$
Portfolio 2 (P2)	Scenario D	-1	0	$PD_{t,T}$
	Scenario ND	-1	$e^{r(T-t)}$	$1-PD_{t,T}$
P1 + P2	Scenario D	0	$e^{r(T-t)}$	$PD_{t,T}$
	Scenario ND	0	$e^{r(T-t)}-e^{r^A(T-t)}$	$1-PD_{t,T}$

Because of the No-Arbitrage assuption, in average the value of P1+P2 is equal to 0:

$$e^{r(T-t)} = PD_{t,T} \times 0 + (1 - PD_{t,T}) \times 1 \times e^{r^A(T-t)}$$

Proof – Implied default probability

$$
\begin{aligned}
e^{r(T-t)} &= PD_{t,T} \times 0 + (1 - PD_{t,T}) \times 1 \times e^{r^A(T-t)} \\
\frac{1}{B(t,T)} &= \mathbb{Q}(\tau > T \mid \tau > t) \frac{1}{\bar{B}^A(t,T)} \\
\mathbb{Q}(\tau > T \mid \tau > t) &= \frac{\bar{B}^A(t,T)}{B(t,T)}
\end{aligned}
$$

as:

- $1 - PD_{t,T} = \mathbb{Q}(\tau > T \mid \tau > t)$
- $B(t,T) = e^{-r(T-t)}$
- $\bar{B}^A(t,T) = e^{-r^A(T-t)}$

▸ Theorem

Proof – Implied default probability with recovery rate strictly superior to 0

Let us assume we have two portfolios, P1 and P2. We take a look at the investment strategy that consists in buying P1 and short-selling P2. Let $PD_{t,T}$ be the probability of default of the considered risky bond between t and T.

Portfolio	Scenarios	In t	Maturity (T)	Probability of the scenario
Portfolio 1 (P1)	Scenario 1	-1	$e^{r(T-t)}$	$PD_{t,T}$
	Scenario 2	-1	$e^{r(T-t)}$	$1 - PD_{t,T}$
Portfolio 2 (P2)	Scenario 1	-1	$e^{r^A(T-t)}R$	$PD_{t,T}$
	Scenario 2	-1	$e^{r^A(T-t)}$	$1 - PD_{t,T}$
P1 - P2	Scenario 1	0	$e^{r(T-t)} - e^{r^A(T-t)}R$	$PD_{t,T}$
	Scenario 2	0	$e^{r(T-t)} - e^{r^A(T-t)}$	$1 - PD_{t,T}$

Given the fact that the initial payment in t is the same in both case and the absence of arbitrage assumption, we have:

$$e^{r(T-t)} = PD_{t,T} \times RPD_{t,T}e^{r^A(T-t)} + (1 - PD_{t,T}) \times 1 \times e^{r^A(T-t)}$$

Proof – Implied default probability with recovery rate strictly superior to 0

Let $PD_{t,T}$ be the probability of default of the considered risky bond between t and T.

$$e^{r(T-t)} = PD_{t,T} \times RPD_{t,T} e^{r^A(T-t)} + (1 - PD_{t,T}) \times 1 \times e^{r^A(T-t)}$$

$$e^{r(T-t)} = PD_{t,T}(Re^{r^A(T-t)} - e^{r^A(T-t)}) + e^{r^A(T-t)}$$

$$PD_{t,T} = \frac{e^{r(T-t)} - e^{r^A(T-t)}}{e^{r^A(T-t)}R - e^{r^A(T-t)}}$$

$$PD_{t,T} = \frac{\frac{e^{r(T-t)}}{e^{r^A(T-t)}} - 1}{R - 1}$$

$$PD_{t,T} = \frac{\frac{\bar{B}^A(t,T)}{B(t,T)} - 1}{R - 1}$$

Proof – Implied default probability with recovery rate strictly superior to 0

as:

- $B(t, T) = e^{-r(T-t)}\bar{B}^A(t, T) = e^{-r^A(T-t)}$ ▸ Theorem

Proof – Survival function in a reduced-form model

We assume that, $\lambda \in \mathbb{R}$ and $\mathbb{Q}(\tau < t + dt \mid \tau > t) = \lambda t$.
We have:

$$
\begin{aligned}
\forall t \in \mathbb{R}^+, \lambda t &= \mathrm{Q}(\tau < t + dt \mid \tau > t) \\
&= \frac{\mathbb{Q}(\tau < t + dt \cap \tau > t)}{\mathbb{Q}(\tau > t)} \text{ (Bayes Formula)} \\
&= \frac{\mathbb{Q}(\tau < t + dt) - \mathbb{Q}(\tau > t)}{\mathbb{Q}(\tau > t)} \text{ TBF} \\
&= \frac{S(t) - S(t + \mathrm{d}t)}{S(t)}
\end{aligned}
$$

Proof – Survival function in a reduced-form model

Thus, we have:

$$\lambda S(t) = \frac{S(t) - S(t + \mathrm{d}t)}{\mathrm{d}t}$$

$\forall t \in \mathbb{R}^+$, when $\mathrm{d}t$ tends to 0, the last fraction tends to $\lambda \times S(t)$. Thus S is differentiable and follows the following differential equation:

$$-\lambda y = y'$$

The solutions of this differential equation have the following expression: $\forall t \in \mathbb{R}, S(t) = Ke^{-\lambda t}$, with $K \in \mathbb{R}$. As S is a survival function of a random variable with support $\mathbb{R}^+$, we have that $S(0) = 1 = K \times e^{-\lambda \times 0} = K$.
We then have, $\forall t \in \mathbb{R}^+, S(t) = e^{-\lambda t}$. Thus, in a reduced-form model with constant intensity of default, the time of default follows an Exponential distribution of parameter λ: $\tau \hookrightarrow \mathcal{E}(\lambda)$

Theorem

Proof – CDS pricing

Fixed leg of a CDS

$$\begin{aligned}
\text{Fixed}(0,T) &= \mathbb{E}^{\mathbb{Q}}\left(s(0,T)\int_0^{\tau\wedge T} e^{-rT}\mathrm{d}t\right)\\
&= s(0,T)\mathbb{E}^{\mathbb{Q}}\left(\int_0^T \mathbb{1}_{\{t<\tau\}}e^{-rt}\mathrm{d}t\right)\\
&= s(0,T)\int_{\mathbb{R}^+}\int_0^T \mathbb{1}_{\{t<h\}}e^{-rt}\mathrm{d}t\; e^{-\lambda h}\mathrm{d}h \qquad \text{(Transfer theorem)}\\
&= s(0,T)\int_0^T\int_{\mathbb{R}^+} \mathbb{1}_{\{t<h\}}e^{-\lambda h}\mathrm{d}h\; e^{-rt}\mathrm{d}t \qquad \text{(Fubini's theorem)}\\
&= s(0,T)\int_0^T\int_t^{+\infty} e^{-\lambda h}\mathrm{d}h\; e^{-rt}\mathrm{d}t\\
&= s(0,T)\int_0^T e^{-(r+\lambda)T}\mathrm{d}t\\
&= s(0,T)\frac{1-e^{-(r+\lambda)T}}{r+\lambda}
\end{aligned}$$

Proof – CDS pricing

Float leg of a CDS

$$
\begin{aligned}
\text{Float}(0,T) &= \mathbb{E}^{\mathbb{Q}}\left((1-R)e^{-r\tau}\mathbb{1}_{\{\tau<t\}}\right) \\
&= (1-R)\int_{\mathbb{R}^+}\lambda e^{-(r+\lambda)h}\mathbb{1}_{\{h<t\}}\mathrm{d}h \qquad \text{(Transfer Theorem)} \\
&= (1-R)\int_0^t \lambda e^{-(r+\lambda)h}\mathrm{d}h \\
&= (1-R)\lambda\frac{1-e^{-(r+\lambda)}}{\lambda+r}
\end{aligned}
$$

▸ Theorem

Proof – Risky duration interpretation

Risky duration interpretation

$$
\begin{aligned}
MtM(t,T) &= Fixed(t,T) - Float(t,T) \\
&= \frac{1-e^{-(r+\lambda)}}{\lambda + r}\left(s(t,T) - (1-R)\lambda\right) \\
&= DV(t,T,\lambda)\left(s(t,T) - (1-R)\lambda\right)
\end{aligned}
$$

Thus we have:

$$
\frac{\mathrm{d}MtM(t,T)}{\mathrm{d}s(t,T)} = DV(t,T,\lambda)
$$

▸ Theorem