Credit Risk
Lecture 1 – Introduction, reduced-form models and CDS

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Class structure and assignments
Class structure and assignments

All the information is on the website:

defaultrisk.free.fr

Credit Risk

École Nationale des Ponts et Chaussées
Département Ingénierie Mathématique et Informatique (IMI) – Master II

This course is part of the training cycle of École Nationale des Ponts et Chaussées: it is a Master II course from the Département Ingénierie Mathématique et Informatique (IMI).

Teaching team, syllabus, grading, access to the forum, etc.
Class structure and assignments
Pedagogical tools (I/II)

Theory
- Infography
- Cheat Sheet
- Slides
- References

Practice
- Quizzes/Flipcards
- Articles and papers
- R/Python notebooks
- Tutorial
- Project
- Case Study

Communication
- Forum
- Questions in class
- Office Hours

Theory
- Concise
- Detailed

Practice
- Individual
- Groups

Communication
- Whenever
- When scheduled
Class structure and assignments
Pedagogical tools (II/II)

<table>
<thead>
<tr>
<th>Buttons</th>
<th>Blocks</th>
<th>Other styles</th>
</tr>
</thead>
</table>
| 🔄 Quit | ❌ Definition - Math
I use this block for math definitions. | ❌ This is a proof. |
| ❌ R Markdown | ❌ Definition - Eco or finance
I use this one for economics or finance definition. | ❌ This is the solution of an exercise, or details of an explanation. |
| ❌ Tutorial | ❌ Be careful
I use this one to draw your attention. | ❌ I am citing: [Harrison and Kreps, 1979]. |
| 🔴 Newspaper | ❌ Example
I use this one when giving concrete examples. | ❌ I will emphasize important words. |
| 🔴 Be Careful | | |
| 🔴 Theorem | | |
| 🔴 Definition | | |

Loïc BRIN, Benoit ROGER
Credit Risk - Lecture 1
Objectives of the lecture

Teaching objectives

At the end of this lecture, you will:

▶ Understand why credit risk is at the basis of our economies;
▶ Have a clear view on the credit risk modeling challenges and outcomes;
▶ Know the basic concepts of credit risk, that is, how to price a bond, what a spread is and how to extract it from the price of a bond, what are the Exposure At Default (EAD), Loss Given Default (LGD) and Probability of Default (PD);
▶ Know what reduced-form models are and how to calibrate them;
▶ Know what Credit Default Swaps are and how to price them.
1. Credit risk and economics

2. Main credit risk outcomes and challenges

3. The basics of credit risk

4. Reduced-form models

5. The Credit Default Swap (CDS), a single-name derivative
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1 Credit risk and economics
   ▶ Credit risk and economics

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5 The Credit Default Swap (CDS), a single-name derivative
Credit risk and economics

What is credit risk?

**credo**: I believe (latin)

**resecare**: To break (latin)

Credit risk – Definition

Credit risk is the risk of **default** on a debt, that may arise from a borrower failing to make required payments. [BCBS, 2000]

Origin of word **bank**

[Fergusson, 2008] is an interesting reference to tackle the subject from an historical perspective.

YouTube
Credit risk and economics

Why is there credit risk?

There is a discrepancy of financial needs among economic agents. Some agents need money to fulfill their projects (firms, states, people, etc.) and other do not need an immediate access to their wealth.

To fill this gap, lenders lend to borrowers, based on the belief that they will retrieve their money and get adequate reward.

This belief – this trust – is at the origin of credit risk.
Credit risk and economics

Who finances the economy?

**Source:** Aspects of Global Asset Allocation, IMF. and own cross-checkings.
Credit risk and economics

Who borrows?

Corporate bonds 86 TUSD

Equity 69 TUSD

Loans 76 TUSD

Public debt securities 58 TUSD

Credit risk and economics

In what banks differ from other lenders?

They have an **expertise** and a defined economic purpose as financial intermediaries.

- They have an expertise in **maturity transformation** (ALM\(^1\) department);
- They have much more **information** on the economy and on their counterparties than any other agent;
- They know how to dissociate risks and underlying assets thanks to **derivative products**;
- They can deal with credit risk on a **macro level** (portfolio approach, dynamic management of assets, macro hedging strategy);
- They **create money** when allowing credits.

See several references at the end of the slides.

---

\(^1\) Asset Liability Management.
Banks are at the center of our economies
- Finance the economy;
- Provide way to transfer risks;
- At the basis of money creation.

Banks face numerous and complex risks
- Credit Risk (~80 %*);
- Market Risk (~10 %*);
- Liquidity Risk;
- Operational Risk (~10 %*).

* Computed as a percentage of Risk Weighted Assets (RWA, see Lecture 5).

It is thus a highly regulated sector
Banks must set capital apart to face an unlikely rise of these risks.

Incentives to reduce balance sheet size

Moving towards an originate to distribute model? (vs originate to hold)
Credit risk and economics

Conclusion
Credit risk and economics

- Credit risk is the risk that a borrower fails to make required payments or has a higher risk not to repay;
- There is no financing of the economy without credit risk;
- Banks finance a big chunk of the economy and are thus prone to credit risk.
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   ▶ The outcomes we will face in this class
   ▶ Real World and Risk Neutral probability
   ▶ Credit risk modeling and the challenge of data

3 The basics of credit risk

4 Reduced-form models

5 The Credit Default Swap (CDS), a single-name derivative
The outcomes we will face in this class

Very different outcomes in comparison with market risk

<table>
<thead>
<tr>
<th>Market Risk</th>
<th>Credit Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount of data</td>
<td>A lot</td>
</tr>
<tr>
<td>Liquidity of the assets</td>
<td>Liquid</td>
</tr>
<tr>
<td>Shape of the loss function</td>
<td>Symmetric</td>
</tr>
<tr>
<td>Correlations</td>
<td>High</td>
</tr>
<tr>
<td>Risk Management</td>
<td>Hedging</td>
</tr>
<tr>
<td>Backtesting</td>
<td>Possible</td>
</tr>
</tbody>
</table>
The outcomes we will face in this class

Let us take a closer look at the two latter.
Let $S_t$ be the variable equals to $s$, if the future state of the economy, in $t$, is $s$.

**Real World Probability, $P$**

Probability that an event, $s$, occurs.

**Risk Neutral Probability, $Q$**

Probability measure which weights the future state of the economy, $s$, according to the price to be risk neutral to that specific state, proportionally to the price to be risk neutral to all the future states of the economy.

This formalization was made by [Harrison and Kreps, 1979].
A simplified example – Real World and Risk neutral probability

Let us say that a future state of the economy (in 3 years) will occur with a (Real World) probability of 10%. Let us suppose that:

- to be sure (i.e. to be risk neutral) to have a cash flow of 1 EUR in 3 years (that is the price of a risk-free zero-coupon bond) costs 0.8 EUR;
- to be sure (i.e. to be risk neutral) to have a cash flow of 1 EUR, only if, the specific state $s$ of the economy occurs in 3 years, costs 0.1 EUR.

We have that: $Q(S_t = s) = \frac{0.1}{0.8} = 12.5\%$ even if $P(S_t = s) = 10\%$.

In that case, the cost of protecting oneself against $s$ is higher than suggested by the Real World probability.
Real World and Risk Neutral probability

Risk Neutral Default rates vs Real World Default rates

<table>
<thead>
<tr>
<th>Rating (in %)</th>
<th>Real World Default rate</th>
<th>Risk Neutral Default rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.03</td>
<td>0.60</td>
</tr>
<tr>
<td>AA</td>
<td>0.06</td>
<td>0.73</td>
</tr>
<tr>
<td>A</td>
<td>0.18</td>
<td>1.15</td>
</tr>
<tr>
<td>BBB</td>
<td>0.44</td>
<td>2.13</td>
</tr>
<tr>
<td>BB</td>
<td>2.23</td>
<td>4.67</td>
</tr>
<tr>
<td>B</td>
<td>6.09</td>
<td>8.02</td>
</tr>
<tr>
<td>CCC</td>
<td>13.52</td>
<td>18.39</td>
</tr>
</tbody>
</table>

Risk Neutral Default rates are higher than Real World Default rates.

That could be because of:

- The **lack of liquidity** on the debt market;
- The **lack of information of investors** on the market;
- The **risk aversion of investors** on the market, etc.
Credit risk modeling and the challenge of data

Where to find data for credit risk modeling?

- Financial Reports
- Rating agencies
- Banks knowledge

- Financial markets prices (bonds, equity, CDS, etc.)

Data to estimate with real world probability

Data to estimate with risk neutral probability
Conclusion
Main credit risk modeling outcomes and challenges

▶ Contrary to market risk, credit risk faces the following challenges: there is few data, the market is illiquid, the loss functions are asymmetric, correlations are low and backtesting is hardly possible;

▶ Additionally, credit risk can be envisioned in many different ways: on several time spans, with a real-world or risk neutral approach, in a continuous or binary perspective, etc. making this risk particularly technical.

▶ may encompass more expert view to drive credit risk assessment
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   ▶ Three points of attention – Discrete version, recovery and risk-free rate
   ▶ The general framework of credit risk modeling: PD, EAD and LGD

4 Reduced-form models

5 The Credit Default Swap (CDS), a single-name derivative
Pricing of bonds – Continuous version

**The spread of a bond**

**Pricing a bond – Continuous version**

Let us denote the continuous and constant coupon rate by $c$, the risky rate of the firm A by $r^A$, the maturity of the bond $T$, and assume the nominal, $N$, is equal to 1, **the price of a bond** is:

$$\bar{B}^A(0, c, T) = 1 + (c - r^A) \frac{1 - e^{-r^A T}}{r^A}$$

The formula is a simple result consequent to the no-arbitrage assumption and the integral resolution of:

$$\bar{B}^A(0, c, T) = \int_0^T ce^{-r^A t} dt + Ne^{-r^A T}$$

**The spread of a bond – Continuous version**

For a given risk-free rate, $r$, the spread of a bond of price $\bar{B}^A(0, T)$, is the value $s^A$ so that:

$$\bar{B}^A(0, c, T) = 1 + (c - (r + s^A)) \frac{1 - e^{-(r+s^A)T}}{(r + s^A)}$$
Pricing of bonds
Pricing and implied survival probability

The implied survival probability – Computed with prices

Let $\tau$ be the time of default of firm A. Let $\bar{B}^A(t, 0, T) = \bar{B}^A(t, T)$ be the price of a zero-coupon risky bond of firm A, at $t$, of maturity $T$, and nominal $N$. Let $B(t, 0, T) = B(t, T)$ be the price of a zero-coupon risk-free bond, at $t$, of maturity $T$, and nominal $N$.

The implied survival probability of firm A, in $T$, from $t$, is:

$$Q(\tau > T \mid \tau > t) = \frac{\bar{B}^A(t, T)}{B(t, T)}$$

It is a consequence of the no-arbitrage assumption.

The implied survival probability – Computed with constant continuous spreads

Let $s^A$ be the spread of Firm A. The implied survival probability can be written:

$$Q(\tau > T \mid \tau > t) = e^{-s^A(T-t)}$$

$$Q(T > \tau \mid \tau > t) = \frac{Ne^{-r(T-t)}}{Ne^{-(r+s^A)(T-t)}} = e^{-s^A(T-t)}$$
Pricing a bond – Discrete version

Let $N$ be the nominal of a bond from firm $A$, $t_1, ..., t_n = T$ the dates when the coupons $C$ are paid, $r_1^A, ..., r_n^A$ the respective risky rates and, $T$, its maturity. The **price of the bond** $\bar{B}^A(0, c, T)$, in 0, is:

$$\bar{B}^A(0, c, T) = \sum_{i=1}^{n} \frac{C}{(1 + r_i^A)^t_i} + \frac{N}{(1 + r_T^A)^T}$$

The spread of a bond – Discrete version

Let $\bar{B}^A(0, c, T)$ be the price of a bond and $r_1, ..., r_n$ the risk-free rates in $t_1, ..., t_n$. The **spread** of $A$ is $s^A$ so that:

$$\bar{B}^A(0, T) = \sum_{i=1}^{n} \frac{C}{(1 + r_i + s^A)^t_i} + \frac{N}{(1 + r_n + s^A)^T}$$
Bootstrap technique to compute implied survival probability – Discrete version

Suppose firm A has \( n \) bonds, \( B_1^A, ..., B_n^A \), paying coupons, \( c \), in \( t_1, ..., t_n \) and of maturity, respectively \( t_1, ..., t_n \). **Bootstrap** works the following way:

- With \( B_1^A \), it is possible to compute \( r_1^A \);
- With \( B_2^A \), subtracting its cash-flow in \( t_1 \) discounted by \( 1 + r_1^A \), it is possible to compute \( r_2^A \);
- etc.

That way, one can compute iteratively \( r_1^A, r_2^A, ..., r_n^A \) and thus deduce \( s_1^A, s_2^A, ..., s_n^A \) using the risk-free rates.
Three points of attention – Discrete version, recovery and risk-free rate

Continuous vs Discrete (III/IV)

Bond clean and dirty prices

Discrete payments implies discontinuity in bond valuation each time a coupon is paid: these discontinuous prices are called **dirty prices**. On the markets, the prices do not suffer such a problem as the so-called **clean prices** are quoted. The formula that links both is:

Dirty price = Clean price + Accrued interests

where Accrued interests = \[
\frac{\# \text{ days since last coupon}}{\# \text{ days between coupons}} \times \text{Coupon rate} \times \text{Nominal}
\]
Adobe Systems Inc. bond valuation

Adobe Systems Inc. (NASDAQ: ADBE) has 600 MUSD worth of bond payable outstanding. The 1 000 USD par, 3.25% semi-annual coupon bonds are due to mature on 1st February 2015. The coupon dates are 1st February and 1st August. They follow 30/360 day count convention and next coupon is due on 1st August 2013. Yvonne Barnet bought 1 000 such bonds from Charles Schwab on 20th July 2013. The market requires buyer to compensate seller for the accrued interest. How much Yvonne must pay Charles? Yvonne must pay the dirty price, but she only knows the clean price: 1036.10 USD.

- days between the transaction date and next coupon date = 11 = 10 days of July plus 1 day of August;
- days in the coupon period = 180 (since 30/360 day count convention is used).

Thus, the dirty price is: 

$$1036.10 + \frac{1000 \times 3.25\%}{2} \times \frac{169}{180} = 1051.36 \text{ USD}$$
Three points of attention – Discrete version, recovery and risk-free rate

The importance of the recovery rate

To simplify math formulas, the recovery rate – the extent to which principal and accrued interests on defaulted debt can be recovered, expressed as a percentage of face value – is often forgotten (or equivalently supposed equal to one). Nonetheless, in practice, the recovery rate must be taken into account when extracting the probability of default from a market price.

The implied probability of default taking into account recovery

Let $R$ be the recovery rate, the implied probability of default taking into account recovery, from now, to $T$, is:

$$PD_{0,T} = \frac{1 - \tilde{B}(0,T)}{B(0,T)} \cdot \frac{1}{1 - R}$$

This is a consequence of the no-arbitrage hypothesis.
What is the risk-free rate?

The rates used as a risk-free rates evolved during the last decades:

- from the government rates, first;
- to the LIBOR rates, then;
- to the Overnight Indexed Swap (OIS) rate, now.
The general framework of credit risk modeling: PD, EAD and LGD

Definitions of PD and EAD

**Probability of Default – PD**

It is the *one-year default probability* of a counterparty.

**Exposure At Default – EAD**

Exposure At Default is the *loss* a bank would suffer if its counterparty defaults, and there would be no guarantee.

This value can either be *known* (loans for example) or *unknown* (lines of credit for instance).
The general framework of credit risk modeling: PD, EAD and LGD

Focus on the Loss Given Default (I/II)

**Loss Given Default – LGD**

Let $R$ be the *recovery rate*, that is the percentage of the exposure recovered by the bank after the default, we have:

\[ \text{LGD} = 1 - R \]

**Loss Given Default and $R$ depend on the underlying credit contract**

<table>
<thead>
<tr>
<th>Contracts</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bank loans</td>
<td>80.3 %</td>
</tr>
<tr>
<td>Senior secured bonds</td>
<td>63.5 %</td>
</tr>
<tr>
<td>Senior unsecured bonds</td>
<td>49.2 %</td>
</tr>
<tr>
<td>Senior subordinated bonds</td>
<td>29.4 %</td>
</tr>
<tr>
<td>Subordinated bonds</td>
<td>29.5 %</td>
</tr>
<tr>
<td>Junior subordinated</td>
<td>18.4 %</td>
</tr>
</tbody>
</table>

*Source*: Moody’s statistics.
The general framework of credit risk modeling: PD, EAD and LGD

Focus on the Loss Given Default (II/II)

Modeling LGD with a beta distribution using a Maximum Likelihood estimator

Let $\alpha > 0$, $\beta > 0$. The density of a beta distribution is:

$$f(x; \alpha, \beta) = \begin{cases} 
  x^{\alpha-1}(1-x)^{\beta-1} & \text{for } x \in [0, 1] \\
  0 & \text{else}
\end{cases}$$

From data, it appears that the shape of LGD distributions is usually a U-shaped curve.
The general framework of credit risk modeling: PD, EAD and LGD

The Expected Loss – EL

We define the Expected Loss as (on time horizon $T$):

$$EL = \mathbb{E}(EAD \times 1_{\{\tau<T\}} \times LGD)$$

$$= \mathbb{E}(EAD) \times \mathbb{E}(1_{\{\tau<M\}}) \times \mathbb{E}(LGD)$$

assump. ⊥

PD

Be Careful!

The independence of EAD, PD and LGD

There is no reason why one should assume independence between EAD, PD and LGD. Actually, some phenomena and papers proved the contrary:

- The phenomenon of gambling for resurrection;
- The study by [Frye, 2013] postulates that LGD is a function of the probability of default.
Pricing of bonds, in a continuous or discrete framework, is based on the **no-arbitrage assumption**;

**Coupons and recovery** are complexities that need to be taken into account when implying probabilities of default from bond prices through respectively accrued interests cleaning and bootstrapping;

There are **three key components** of any credit expected losses estimator, the Exposure At Default, the Loss Given Default and the Probability of Default – their independence, while often assumed in models, is not always tenable;

The **Probability of Default (PD)** is the major focus of this course.
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   ▶ Reduced-form models
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Reduced-form models

Definition

Reduced-form models consist in modeling the conditional law of the random time of default.

Conditional default probability

In a reduced-form model, conditional default probability is defined as:

\[ \mathbb{Q}(\tau < t + dt \mid \tau > t) = \lambda dt \]

\( \lambda \) is the default intensity and corresponds to the instantaneous forward default rate. This variable is exogenous to the problem.

Survival function

In a reduced-form model, with \( \lambda \) constant, the survival function is defined as:

\[ S(t) = \mathbb{Q}(\tau > t) = \exp(-\lambda t) \]

\( \tau \), the the time of default, follows a Geometric distribution of parameter \( \lambda \).
Reduced-form models

What is default intensity?

Is the default intensity, $\lambda$, constant or stochastic?

It depends:

- **Constant**: Time homogeneous Poisson Process;
- **Deterministic**: Time deterministic inhomogeneous Poisson Process;
- **Stochastic**: Time-varying and stochastic Poisson Process as the Cox, Ingersoll, Ross (CIR) model.
The implied survival probability

Let \( B(0, t) \) be a zero-coupon risk-free bond and \( \bar{B}(0, t) \) be a risky zero-coupon bond. We have:

\[
Q(\tau > t) = \frac{\bar{B}(0, t)}{B(0, t)}
\]

This is a result based on the no-arbitrage assumption.

The implied survival probability – Application for calibration of intensity models

We deduce from the above formula the expression of the default intensity, \( \lambda \):

\[
\lambda = -\frac{\log \left( \frac{\bar{B}(0, t)}{B(0, t)} \right)}{t}
\]

We have seen that, \( Q(\tau > t) = e^{-\lambda t} \)
Reduced-form models are models based on an **exogenous variable**, called here, the default intensity;

The **default intensity is often assumed constant** but can also be non-constant or even stochastic.
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   - Reduced-form models applied to CDS pricing
   - Default swaptions and other swaps
Credit Default Swap

Credit Default Swaps (CDS) are financial agreements that allows the transfer of the credit risk of a loan to another counterparty.

▶ Bank A holds a loan on corporate C on its balance-sheet and buys protection on the credit risk related to counterparty C;
▶ Bank B sells protection on corporate C and receives the payment of a premium.

Are CDS insurance contracts?

CDS are neither insurance contracts, nor guarantees.

CDS and JP Morgan

JP Morgan pioneered the CDS, more precisely, Blythe Masters is considered the inventor of this security.
Single-name credit derivatives and Credit Default Swap (CDS)

CDS cash flows

- **Bank A**
- **Bank B**
- **Reference C**

**CDS contract**
- Fees
- Protection in case of default

**Bond contract**
- Principal
- Interests and principal

**CDS contract**
Statistics on the CDS market

The CDS market

The derivatives market

<table>
<thead>
<tr>
<th>Category</th>
<th>Market Value (TUSD)</th>
<th>Notional (TUSD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unallocated</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Credit derivatives</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>Commodity</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Equity-linked</td>
<td>10.0</td>
<td>10.0</td>
</tr>
<tr>
<td>Interest Rate</td>
<td>28.3</td>
<td>28.3</td>
</tr>
<tr>
<td>Foreing Exchange</td>
<td>1.1</td>
<td>1.1</td>
</tr>
<tr>
<td>Total</td>
<td>36.8</td>
<td>36.8</td>
</tr>
</tbody>
</table>

| Notional (TUSD) | 68.6 |

| Unallocated       | 3.0  |
| Credit derivatives| 6.1  |
| Total             | 9.1  |

The CDS market

- **Unallocated**: 14% (notional: 1.4 TUSD)
- **Credit derivatives**: 14% (notional: 6.1 TUSD)
- **Commodity**: 21% (notional: 10.0 TUSD)
- **Equity-linked**: 21% (notional: 28.3 TUSD)
- **Interest Rate**: 65% (notional: 36.8 TUSD)
- **Foreing Exchange**: 7% (notional: 9.1 TUSD)

**Market value**: 10.0 TUSD (TUSD)

- **Non-rated**: 14%
- **Investment grade**: 66%
- **Below investment grade**: 21%

- **>5Y**: 7%
- **<5Y and >3Y**: 65%
- **<3Y**: 23%

**Notional (TUSD)**

- **Sovereigns**: 22%
- **Securitized products and multiple sector**: 40%
- **Multi-name**: 43%
- **Non-financial institutions**: 16%
- **Financial Firms**: 25%

**Non-financial institutions**: 2%

- **Insurance**: 2%
- **Banks and securities firms**: 6%
- **Hedge funds**: 6%
- **SPVs**: 14%

**CCPs**: 62%

**Investment grade**: 66%

**Below investment grade**: 21%

**Notional (TUSD)**

- **Single-name**: 57%
- **Non-rated**: 14%

**Credit derivatives**: 67%

**Single-name**: 57%

**Investment grade**: 66%

**Below investment grade**: 21%

**Credit derivatives**: 67%
Legal aspect of CDS

The International Swaps and Derivatives Association (ISDA)

ISDA defines standards for credit derivatives transactions:

- **1999**: definitions;
- **2003**: supplements;
- **2009**: big-bang protocol.
Specifications for a CDS

Examples of important specifications for a CDS

▶ What is a CDS credit event?
  - bankruptcy;
  - failure to pay (coupon or nominal);
  - restructuring of a bond.

▶ Settlement:
  - Physical settlement: the buyer of the CDS gives the defaulted bonds to the seller, and receives the nominal (N);
  - Cash settlement: the buyer receives $N(1 - R)$, and keeps the defaulted bonds.

▶ Normal vs Digital CDS:
  - Digital: the payoff is $N$;
  - Normal: the payoff is $(1 - R)N$. 
Various definitions of default depending on seniority

Definitions of default for financial institutions in Europe after the crisis

The new banking regulations that followed the 2008 to 2011 economic crisis had consequences on the financial ratios that banks should respect. The ISDA adapted its definition of default to embrace the existence of these new ratios.

Several CDS for one company

One company can have different Credit Default Swaps adapted to several maturities and debt seniorities: that is the reason why you will find several CDS references for the same company on The Ice deals of the day page.
Single-name credit derivatives and Credit Default Swap (CDS)

Use of CDS

What CDS can be used for?

CDS are mostly used for risk management and investment strategies.

▶ Risk mitigation;
▶ Management of credit lines;
▶ Bank’s capital management;
▶ Balance sheet management;
▶ Leverage effect (with First to Default product for example or CDO tranches);
▶ Access to the market.

Who uses CDS?

CDS are mostly used by financial institutions and very few corporate firms and retail customers use these securities:

▶ Banks;
▶ Corporate firms;
▶ Insurers and reinsurers;
▶ Asset managers;
▶ Hedge funds.
Reduced-form models applied to CDS pricing

Fixed leg of a CDS

Value of the fixed leg of a CDS

The value of the fixed leg of a CDS is:

\[
\text{Fixed}(0, T) = s(0, T) \frac{1 - e^{-(r+\lambda)T}}{r + \lambda}
\]

Fixed leg pays the reference spread at inception of the CDS up to the minimum of the default date (\(\tau\)) and maturity \(T\).

For a continuous paid spread:

\[
\text{Fixed}(0, T) = \mathbb{E}^{\mathbb{Q}} \left( s(0, T) \int_{0}^{\tau \wedge T} e^{-rt} \, dt \right)
\]

\[
= s(0, T) \int_{0}^{T} e^{-(r+\lambda)t} \, dt
\]

\[
= s(0, T) \frac{1 - e^{-(r+\lambda)T}}{r + \lambda}
\]
Reduced-form models applied to CDS pricing

Floating leg of a CDS

Value of the floating leg of a CDS

The value of the floating leg of a CDS is:

\[
\text{Floating}(0, T) = (1 - R) \frac{\lambda}{\lambda + r} \left( 1 - e^{-(\lambda + r)T} \right)
\]

The floating leg is paid when a default occurs: the protection seller pays the difference between the notional and the recovery.

\[
\text{Floating}(0, T) = E^Q\left( (1 - R)e^{-r\tau} 1_{\{\tau < T\}} \right) = (1 - R) \frac{\lambda}{\lambda + r} \left( 1 - e^{-(\lambda + r)T} \right)
\]
The spread of a CDS

The spread of a CDS is:

\[ s = \lambda (1 - R) \]

At inception, the Net Present Value (NPV) for the protection seller is:

\[ \text{MtM}(0, T) = \text{Fixed}(0, T) - \text{Floating}(0, T) \]

The fair spread sets the initial MtM at 0. We thus have \( s = \lambda (1 - R) \).

MtM through the time and sensitivity of the CDS to the time

The sensitivity of the MtM is the risky duration, DV:

\[ \text{DV}(t, T, \lambda) = \frac{1 - e^{-(r + \lambda)(T-t)}}{r + \lambda} \]
Reduced-form models applied to CDS pricing

CDS sensitivity

**Present Value of a CDS through the time**

Let $s_0$, be the spread in $t = 0$, and $s_t$, the spread today, in $t$.

The **Present Value of the protection seller** is:

$$ PV(s_0, s_t) = DV(0, t, \lambda)(s_0 - s_t) $$
CDS – Discrete vs continuous pricing

Once again, these formulas suppose continuous interest rates payments and spread payments. These formulas are nice proxies to assess CDS spreads, but for precise pricing, one must take into account each flow separately.
Reduced-form models applied to CDS pricing

CDS spread vs Bond spread: the CDS basis

CDS spread and Bond spread

Generally, CDS spreads are larger than Bond spreads.

Several reasons explain this phenomenon:

▶ definition of credit event is different;
▶ the protection buyer has no impact on the bond issuer through covenants.

On the other hand, funds and insurers sell massively protection, making CDS spread tighten.

For more information on the subject, you can take a look at [Choudhry, 2006].
Default swaptions and other swaps

An introduction to default swaptions

Knock-out swaptions

Knock-out swaptions:

▶ Allow to sell or buy protection at maturity;
▶ Payoff is null if default occurs before maturity.

The pricing is done with models similar to Black-Scholes model.
Default swaptions and other swaps

Other swaps

- **Constant Maturity Credit Default Swaps (CMCDS):** is a usual CDS except that the fixed leg of the contract is computed each semester with the new data. Its pay-off is often capped.

- **Loan-CDS (LCDS):** they are linked to a specific loan, not on a specific name; thus, would the loan be reimbursed in advance, the CDS would be cancelled.

- **Forward CDS:** they are CDS that offer protection from a future date, $T$ to a future data $T + M$.

- **Cancellable CDS:** they are CDS that are cancellable at a or several future dates;

- **Total Return Swap (TRS):** they are financial contracts that transfer both the credit and the market risk of an asset against either a variable or a fixed rate (they are in fact like funded Interest Rate Swaps – IRS).
Conclusion

Reduced-form models

- CDS are contacts that protect against default;
- Their valuations reconcile the no-arbitrage assumption and the reduced-form models;
- CDS spreads can differ from bond spreads.
Conclusion

Reduced-form models and Credit Default Swaps (CDS)

▶ To be filled.
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Proofs of the lecture
Proofs of the lecture

- Bond price, continuous case
- Implied default probability
- Implied default probability with recovery rate strictly superior to 0
- Survival function in a reduced-form model
- CDS pricing
- Risky duration interpretation
Proof – Bond price, continuous case

\[ \bar{B}^A(0, c, T) = \int_0^T ce^{-r^A t} \, dt + e^{-r^A T} \]

\[ = \frac{-c}{r^A} \left[ e^{-r^A t} \right]_0^T + e^{-r^A T} \]

\[ = \frac{-c}{r^A} (e^{-r^A T} - 1) + e^{-r^A T} \]

\[ = \frac{c}{r^A} - \frac{ce^{-r^A T}}{r^A} + e^{-r^A T} \]

\[ = \frac{c - r^A}{r^A} + 1 - \frac{ce^{-r^A T} - r^A e^{-r^A T}}{r^A} \]

\[ = \frac{c - r^A}{r^A} (1 - e^{-r^A T}) + 1 \]
Proof – Implied default probability

Let us assume we have two portfolios, P1 and P2. We take a look at the investment strategy that consists in buying P1 and short-selling P2. Let $PD_{t,T}$ be the probability of default of the considered risky bond between $t$ and $T$.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Scenarios</th>
<th>In $t$</th>
<th>Maturity ($T$)</th>
<th>Probability of the scenario</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio 1 (P1)</td>
<td>Scenario 1</td>
<td>-1</td>
<td>$e^{rt}$</td>
<td>$PD_{t,T}$</td>
</tr>
<tr>
<td></td>
<td>Scenario 2</td>
<td>-1</td>
<td>$e^{rT}$</td>
<td>$1 - PD_{t,T}$</td>
</tr>
<tr>
<td>Portfolio 2 (P2)</td>
<td>Scenario 1</td>
<td>-1</td>
<td>0</td>
<td>$PD_{t,T}$</td>
</tr>
<tr>
<td></td>
<td>Scenario 2</td>
<td>-1</td>
<td>$e^{rT}$</td>
<td>$1 - PD_{t,T}$</td>
</tr>
<tr>
<td>P1 - P2</td>
<td>Scenario 1</td>
<td>0</td>
<td>$e^{rt}$</td>
<td>$PD_{t,T}$</td>
</tr>
<tr>
<td></td>
<td>Scenario 2</td>
<td>0</td>
<td>$e^{rT} - e^{rAT}$</td>
<td>$1 - PD_{t,T}$</td>
</tr>
</tbody>
</table>

Given the fact that the initial payment in $t$ is the same in both case and the absence of arbitrage assumption, we have:

$$e^{rt} = PD_{t,T} \times 0 + (1 - PD_{t,T}) \times 1 \times e^{rAT}$$
Proof – Implied default probability

\[ e^{rT} = PD_{t,T} \times 0 + (1 - PD_{t,T}) \times 1 \times e^{rA_T} \]

\[ \frac{1}{B(t, T)} = Q(\tau > T \mid \tau > t) \frac{1}{\bar{B}^A(t, T)} \]

\[ Q(\tau > T \mid \tau > t) = \frac{\bar{B}^A(t, T)}{B(t, T)} \]

as:

- \(1 - PD_{t,T} = Q(\tau > T \mid \tau > t)\)
- \( B(t, T) = e^{-rt} = \frac{1}{e^{-rt}}\)
- \( \bar{B}^A(t, T) = e^{-rA_t} = \frac{1}{e^{-rA_t}}\)
Proof – Implied default probability with recovery rate strictly superior to 0

Let us assume we have two portfolios, P1 and P2. We take a look at the investment strategy that consists in buying P1 and short-selling P2. Let $PD_{t,T}$ be the probability of default of the considered risky bond between $t$ and $T$.

<table>
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<tr>
<td>Portfolio 1 (P1)</td>
<td>Scenario 1</td>
<td>-1</td>
<td>$e^{rT}$</td>
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<tr>
<td></td>
<td>Scenario 2</td>
<td>-1</td>
<td>$e^{rT}$</td>
<td>$1 - PD_{t,T}$</td>
</tr>
<tr>
<td>Portfolio 2 (P2)</td>
<td>Scenario 1</td>
<td>-1</td>
<td>$e^{A_T R}$</td>
<td>$PD_{t,T}$</td>
</tr>
<tr>
<td></td>
<td>Scenario 2</td>
<td>-1</td>
<td>$e^{A_T}$</td>
<td>$1 - PD_{t,T}$</td>
</tr>
<tr>
<td>P1 - P2</td>
<td>Scenario 1</td>
<td>0</td>
<td>$e^{rT} - e^{A_T R}$</td>
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</tr>
<tr>
<td></td>
<td>Scenario 2</td>
<td>0</td>
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<td>$1 - PD_{t,T}$</td>
</tr>
</tbody>
</table>

Given the fact that the initial payment in $t$ is the same in both case and the absence of arbitrage assumption, we have:

$$e^{rT} = PD_{t,T} \times RPD_{t,T} e^{A_T} + (1 - PD_{t,T}) \times 1 \times e^{A_T}$$
Proof – Implied default probability with recovery rate strictly superior to 0

Let $PD_{t,T}$ be the probability of default of the considered risky bond between $t$ and $T$.

$$e^{rT} = PD_{t,T} \times RPD_{t,T} e^{r_A T} + (1 - PD_{t,T}) \times 1 \times e^{r_A T}$$

$$e^{rT} = PD_{t,T}(Re^{r_A T} - e^{r_A T}) + e^{r_A T}$$

$$PD_{t,T} = \frac{e^{rT} - e^{r_A T}}{e^{r_A T} R - e^{r_A T}}$$

$$PD_{t,T} = \frac{e^{rT}}{e^{r_A T}} - \frac{1}{R - 1}$$

$$PD_{t,T} = \frac{\bar{B}^A(t,T)}{B(t,T)} - \frac{1}{R - 1}$$
Proof – Implied default probability with recovery rate strictly superior to 0

\[ B(t, T) = e^{-rt} = \frac{1}{e^{-rt}} \]

\[ \bar{B}^A(t, T) = e^{-r^A t} = \frac{1}{e^{-r^A t}} \]
Proof – Survival function in a reduced-form model

We assume that, $\lambda \in \mathbb{R}$ and $Q(\tau < t + dt \mid \tau > t) = \lambda t$.

We have:

\[
\forall t \in \mathbb{R}^+, \lambda t = Q(\tau < t + dt \mid \tau > t) = \frac{Q(\tau < t + dt \cap \tau > t)}{Q(\tau > t)} \quad \text{(Bayes Formula)}
\]

\[
= \frac{Q(\tau < t + dt) - Q(\tau > t)}{Q(\tau > t)} = \frac{S(t + dt) - S(t)}{S(t)} \quad \text{TBF}
\]
Proof – Survival function in a reduced-form model

Thus, we have:

\[ \lambda S(t) = \frac{S(t + dt) - S(t)}{dt} \]

\( \forall t \in \mathbb{R}^+ \), when \( dt \) tends to 0, the last fraction tends to \( \lambda \times S(t) \). Thus \( S \) is differentiable and follows the following differential equation:

\[ \lambda y = y' \]

The solutions of this differential equation have the following expression:
\( \forall t \in \mathbb{R}, S(t) = Ke^{-\lambda t} \), with \( K \in \mathbb{R} \). As \( S \) is a survival function of a random variable with support \( \mathbb{R}^+ \), we have that \( S(0) = 1 = K \times e^{-\lambda \times 0} = K \).

We then have, \( \forall t \in \mathbb{R}^+, S(t) = e^{-\lambda t} \). Thus, in a reduced-form model with constant intensity of default, the time of default follows a geometric distribution of parameter \( \lambda \): \( \tau \mapsto \mathcal{G}(\lambda) \)
Proof – CDS pricing

Fixed leg of a CDS

\[
\text{Fixed}(0, T) = \mathbb{E}^Q \left( s(0, T) \int_0^{\tau \wedge T} e^{-rT} \, dt \right)
\]

\[
= s(0, T) \mathbb{E}^Q \left( \int_0^T \mathbbm{1}_{\{t < \tau\}} e^{-rt} \, dt \right)
\]

\[
= s(0, T) \int_{\mathbb{R}^+} \int_0^T \mathbbm{1}_{\{t < h\}} e^{-rt} \, dt \, e^{-\lambda h} \, dh \quad \text{(Transfer theorem)}
\]

\[
= s(0, T) \int_0^T \int_{\mathbb{R}^+} \mathbbm{1}_{\{t < h\}} e^{-\lambda h} \, dh \, e^{-rt} \, dt \quad \text{(Fubini’s theorem)}
\]

\[
= s(0, T) \int_0^T \int_t^{+\infty} e^{-\lambda h} \, dh \, e^{-rt} \, dt
\]

\[
= s(0, T) \int_0^T e^{-(r+\lambda)t} \, dt
\]

\[
= s(0, T) \frac{1 - e^{-(r+\lambda)T}}{r + \lambda}
\]
Proof – CDS pricing
Float leg of a CDS

\[ \text{Float}(0, T) = \mathbb{E}^Q ((1 - R)e^{-r\tau} \mathbb{1}_{\{\tau < t\}}) \]

\[ = (1 - R) \int_{\mathbb{R}^+} \lambda e^{-(r+\lambda)h} \mathbb{1}_{\{h < t\}} \, dh \] (Transfer Theorem)

\[ = (1 - R) \int_0^t \lambda e^{-(r+\lambda)h} \, dh \]

\[ = (1 - R) \lambda \frac{1 - e^{-(r+\lambda)}}{\lambda + r} \]
Proof – Risky duration interpretation

Risky duration interpretation

\[ MtM(t, T) = Fixed(t, T) - Float(t, T) = 1 - e^{-(r+\lambda)} \frac{\lambda + r}{\lambda + r}(s(t, T) - (1 - R)\lambda) = DV(t, T, \lambda)(s(t, T) - (1 - R)\lambda) \]

Thus we have:

\[ \frac{dMtM(t, T)}{ds(t, T)} = DV(t, T, \lambda) \]