Credit Risk

Lecture 2 – Structural models: Merton and Leland models

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Objectives of the lecture

Teaching objectives

At the end of this lecture, you will:

- Understand the **principles of structural approaches** in credit risk;
- Know how to **compute the equity and debt values of a firm** under the Merton model's assumptions;
- Be able to compute the **Merton probability of default** of a firm;
- Know how to derive the **optimal amount of debt** for a firm's investors from the Leland model;
- Be aware of the **limitations of structural approaches**.

1 Necessary prerequisites for structural models

2 The Merton model

3 The Leland model

Table of Contents

1 Necessary prerequisites for structural models
- Accounting basics
- Option theory basics

2 The Merton model

3 The Leland model

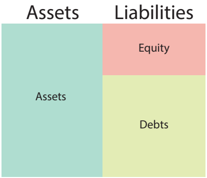
Accounting basics

Balance sheet structure

Firms finance their Assets with Liabilities:

- Equity ;
- Debts.

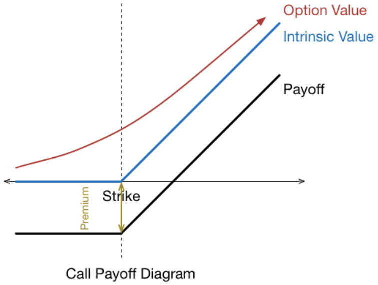
All of this is summarized in their balance sheet.



Structural model are based on the structure of the liabilities of the firm.

Option theory basics

Call option payoff



Source: www.brilliant.org/wiki/call-and-put-options/

Option theory basics

Put option payoff

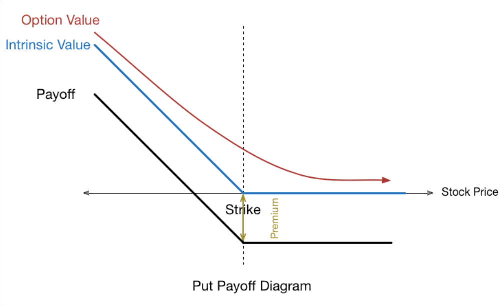


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Table of Contents

1 Necessary prerequisites for structural models

2 The Merton model
- Merton Model – Framework
- Merton model – Main results
- Merton model and beyond

3 The Leland model

Merton Model – Framework

What are the main assumptions?

Balance sheet is equilibrated. If the value of assets changes, so do the liabilities. Let us suppose that the debt is a zero-coupon bond of maturity T.

↓

If assets value is inferior to the debt nominal, that is, **if equity is inferior to 0**: the firm is in default.

↓

In case of liquidation, bonds and loans investors expect to recover the nominal of the debt (D), and **equity holders get what remains**.

Value of Assets (V_T)	Shareholder's flow	Debt holders's flow
$V_T \geq D$	$V_T - D$	D
$V_T < D$	0	V_T

Merton Model – Framework

The value of the debt

Under the risk-neutral probability, the debt value at t is equal to the **expected discounted cash flows from the debt** at maturity T:

The debt value of the debt (Merton)

$$
\begin{aligned}
D_t &= \mathbb{E}^Q\left[e^{-r(T-t)} \min\left(D, V_T\right) \mid \mathcal{F}_t\right] \\
&= \underbrace{\mathbb{E}^Q\left[e^{-r(T-t)} D \mid \mathcal{F}_t\right]}_{\text{risk-free ZC value}} - \underbrace{\mathbb{E}^Q\left[e^{-r(T-t)} \left(D - V_T\right)^+ \mid \mathcal{F}_t\right]}_{\text{Put value}}
\end{aligned}
$$

Hence, the value of the **debt** is equal to the **price of a risk-free zero-coupon** of maturity T **minus the value of a put on the value of the assets** of maturity T and strike D.

Merton Model – Framework

The value of the equity

Under the risk-neutral probability, the **equity** value at t is equal to the **expected discounted cash flows of the shareholders**:

The equity value (Merton)

$$E_t = \underbrace{\mathbb{E}^Q \left[e^{-r(T-t)} \max \left(V_T - D, 0 \right) \mid \mathcal{F}_t \right]}_{\text{Call value}}$$

The value of the **equity** is then equal to the price of **call on the firm assets** of maturity T and strike D.

Merton model – Main results

Diffusion of asset's value

Diffusion of Asset's value in Merton model

Let $(V_t)_t$ be the process modeling the value of the firm. In Merton model, we have:

$$\frac{dV_t}{V_t} = r dt + \sigma d\tilde{W}_t$$

where $\tilde{W}_t$ denote a standard brownian motion under the risk-neutral probability an r the risk-free rate.

The **asset's value** of the firm is modelled with a **geometric brownian motion**.

Merton model – Main results

How to get the value of the debt and equity?

Value of debt and equity – Black-Scholes results

Let D be the amount of debt in the balance sheet in t, D_t its value, and E_t the value of equity (in t). We have:

$$D_t = De^{-r(T-t)}\mathcal{N}(d_2) + V_t\mathcal{N}(-d_1)$$

$$E_t = V_t\mathcal{N}(d_1) - De^{-r(T-t)}\mathcal{N}(d_2)$$

with:

$$d_1 = \frac{\log \frac{V_t}{D} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

Since the value of the firm follows a geometric brownian motion, we can derive the **debt and equity values** using the **Black-Scholes** formula.

Merton model – Main results

Can we derive the Debt's spread?

The debt **spread** (s_t) of the debt (D) in t is the **actual interest rate minus the risk-free rate** (r), so that $D_t = De^{-(r+s_t)(T-t)}$.

Spread – Value

In Merton model, the spread is then equal to:

$$s_t = \frac{1}{T-t} \log\left(\frac{D}{D_t}\right) - r$$

where $D_t = De^{-r(T-t)}\mathcal{N}(d_2) + V_t\mathcal{N}(-d_1)$

Merton model – Main results

What does the Call-Put parity mean in Merton model?

Call-Put parity

The Call-Put parity corresponds to an obvious **equation from corporate finance**.

$$\text{Assets} = \text{Equity} + \text{Debt}$$

Reminder: Let A_t denote the value of an asset in t and C_t and P_t denote the value of a European call option and a European put on the underlying A_t with strike D. The **classical call-put parity** is given by:

$$C_t - P_t = A_t - \underbrace{De^{-r(T-t)}}_{risk-freeZCvalue}$$

Tutorial

Merton model – Main results

What are the PD and LGD of the firm?

PD and LGD

From this model, we can compute PD and LGD.

$$\text{PD} = \mathbb{Q}(V_T \leq D) = \mathcal{N}(-d_2)$$

$$\text{LGD} = \mathbb{E}^Q(D - V_T \mid V_T \leq D) = \frac{\mathbb{E}^Q((D - V_T)^+)}{\mathbb{Q}(V_T \leq D)}$$

In the Merton model the **default** occurs when the **firm's (assets) value falls below the nominal of its debt**. The Loss Given Default **(LGD) is the expected value of the firm after the debtors are paid**.

Merton model and beyond

Pros and cons

- Pros :
  - **Economic interpretation**.
- Cons:
  - There is no conclusion on the **optimal amount of the debt**;
  - The model is very **bad for short term default** probability;
  - **Debt structure is too simplistic**;
  - **Debt evolution is exogenous**.

Merton model and beyond

Merton model extension

- With **jumps**;
- With a **barrier option approach** [Black et al., 1976];
- With a **stochastic interest rate** [Longstaff et al., 1995];
- With a **barrier option approach and stochastic interest rate** [Brys et al., 1997];
- Taking into account **imperfect information of bond investors** [Duffie et al., 1997];
- With an **endogenous debt** (see next slides).

Quiz | Notebook | Tutorial

Conclusion

The Merton model

- To be filled.

Table of Contents

1 Necessary prerequisites for structural models

2 The Merton model

3 The Leland model
- Leland Model – Framework
- Merton model – Main results
- Leland model computations
- Optimal debt and default trigger

Leland Model – Framework

What are the main improvements to Merton model?

Framework of Leland's model

- At $t = 0$, the owners of a debt-free firm decide to issue debt to optimize their equity value.
- There are two control parameters:
  - K the **default trigger**;
  - D_0 the size of the debt.
- There are two other parameters:
  - $\tau \in [0, 1]$ the **tax benefit** gained on debt coupons;
  - $\alpha \in [0, 1[$ the fraction of asset value lost at the time of bankruptcy due to **frictions**.

Leland Model – Framework

What are the main assumptions?

Leland Model

- The firm asset value A_t follows a **Geometric Brownian Motion**:

$$\frac{dA_t}{A_t} = (r - \delta)\,dt + \sigma dW_t^Q$$

where r denotes the risk-free rate and δ the dividend rate
- The debt is a perpetual bond that pays a **constant coupon rate** C every unit of time
- As specified in the contracts (**covenants**), the default of the firm is triggered when $A_t \leq K$:

$$\tau_B = \inf\{t | A_t \leq K\}$$

- Prior to the default we always have $E_t \geq 0$, where E_t denotes the value of the equity at time t
- At $t = \tau_B$, the debt value D_{τ_B} is equal to $(1-\alpha)K$ with $\alpha \in [0, 1[$
- There is a **tax rebate rate** $\tau \in [0, 1]$ on the debt coupons

At $t = 0$, the value of the debt-free company is A_0, but the owners have to **surrender a part of their equity** to collect $D_0 > 0$ so that $E_0 < A_0$. The problem for the owners is then to **maximize the firm value** $v_0 = E_0 + D_0 \geq A_0$.

Merton model – Main results

Model simplifications

Calculation simplifications

As a matter of simplicity:

- we assume that $\mathbb{Q}(\tau_B = \infty) = 0$, but the conclusions would remain the same without this assumptions;
- the value of the firm, its debt and its equity will be computed for $t = 0$ to simplify notations.

Merton model – Main results

The value of the debt

Debt value at t=0

Using the **risk neutral probability** the value of the debt issued at $t = 0$ is equal to the **expected discounted cash flows from this debt**:

$$D_0 = \mathbb{E}^Q \left[e^{-r\tau_B} (1-\alpha) K \mathbb{1}_{\{0 \leq \tau_B < \infty\}} \right] + \mathbb{E}^Q \left[\int_0^{\tau_B} C e^{-rt} dt \right] \quad (1)$$

The debt value at $t = 0$ is equal to the expected discounted cash flows from the liquidation value of the assets at $t = \tau_B$ and from the coupons.

Merton model – Main results

The value of the firm

The firm value at t=0

After the issuance of the debt, the firm value is:

$$v_0 = A_0 + \mathbb{E}^Q \left[\int_0^{\tau_B} \tau C e^{-rt} dt \right] - \mathbb{E}^Q \left[e^{-r\tau_B} \alpha K \mathbb{1}_{\{0 \leq \tau_B < \infty\}} \right] \tag{2}$$

After recapitalization the firm value is equal to the expected discounted cash flows from the **tax rebate** on coupons minus those lost from the **friction** when the assets are sold plus the **initial value of the assets**.

Merton model – Main results

The value of the equity

The equity value at t=0

We can deduce the equity value of the firm at $t = 0$ using (1) and (2):

$$E_0 = v_0 - D_0 = \mathbb{E}^Q \left[\int_0^{\tau_B} (\delta A_t - (1-\tau) C) e^{-rt} dt \right] \quad (3)$$

The previous calculation being non-trivial, the equity can also be seen as the expected discounted cash flows from the **dividends** minus those from the fraction of the **coupons** when the tax rebate has been taken into account.

Leland model computations

Laplace transform

Laplace transform of the stopping time

The **Laplace transform** $L(a, b, \mu) = \Phi_{\tau_B}(a)$ for $\tau_B = \inf\{t \mid W_t + \mu t \geq b\}$ is given by:

$$\Phi_{\tau_B} = \mathbb{E}\left[e^{-a\tau_b}\right] = e^{b\left(\mu - \sqrt{\mu^2 + 2a}\right)}$$

Leland model computations

Computation of the Laplace transform of Leland's bankruptcy time

Since A_t is a **Geometric Brownian Motion** we have:

$$A_t = \exp\left(\ln(A_0) + \left(r - \delta - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$$

so that:

$$\mathbb{E}^Q\left[e^{-r\tau_B}\mathbb{1}_{\{0\leq\tau_B<\infty\}}\right] = L(r, d_0, -m) = \left(\frac{A_0}{K}\right)^{-\gamma} \quad (4)$$

where:

$$d_0 = \frac{1}{\sigma}\ln\left(\frac{A_0}{K}\right),$$

$$m = \frac{1}{\sigma}\left(r - \delta - \frac{1}{2}\sigma^2\right) \leq 0,$$

$$\gamma = \frac{1}{\sigma}\left(m + \sqrt{m^2 + 2r}\right) > 0$$

Leland model computations

Computation of the debt value

From (4) we derive that:

$$\mathbb{E}^Q\left[e^{-r\tau_B}(1-\alpha)K\mathbb{1}_{\{0\leq\tau_B<\infty\}}\right] = (1-\alpha)KL(r,d_0,-m) = (1-\alpha)K\left(\frac{A_0}{K}\right)^{-\gamma} \tag{5}$$

and

$$\mathbb{E}^Q\left[\int_0^{\tau_B} Ce^{-rt}dt\right] = \mathbb{E}^Q\left[\frac{C}{r}\left(1-e^{-r\tau_B}\right)\mathbb{1}_{\{0\leq\tau_B<\infty\}} + \frac{C}{r}\mathbb{1}_{\{\tau_B=\infty\}}\right] = \frac{C}{r}\left[1-\left(\frac{A_0}{K}\right)^{-\gamma}\right] \tag{6}$$

From (5) and (6) we have:

Debt value at t=0

$$D_0 = (1-\alpha)K\left(\frac{A_0}{K}\right)^{-\gamma} + \frac{C}{r}\left[1-\left(\frac{A_0}{K}\right)^{-\gamma}\right] \tag{7}$$

Leland model computations

Computation of the firm and equity values

In the fashion as for the debt value, we derive from (4) that:

Firm value at t=0

$$v_0 = A_0 + \frac{C\tau}{r}\left[1 - \left(\frac{A_0}{K}\right)^{-\gamma}\right] - \alpha K \left(\frac{A_0}{K}\right)^{-\gamma} \tag{8}$$

and from (7) and (8) we can conclude that:

Equity value at t=0

$$E_0 = v_0 - D_0 = A_0 - \frac{(1-\tau)\,C}{r}\left[1 - \left(\frac{A_0}{K}\right)^{-\gamma}\right] - K \left(\frac{A_0}{K}\right)^{-\gamma} \tag{9}$$

Optimal debt and default trigger

Are there optimal debt and default trigger for the sharehorlders?

By maximizing (9) we get the **optimal** C and K for the **shareholders**:

How should the owners choose C and K?

$$K^*(C) = \frac{\gamma(1-\tau)C}{(\gamma+1)r}$$
$$C^* = A_0 \frac{(\gamma+1)r}{\gamma(1-\tau)}\left[\frac{(1+\gamma)\tau+\alpha(1-\tau)\gamma}{\tau}\right]^{-\gamma} \tag{10}$$

Quiz Notebook Tutorial

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