Credit Risk

Lecture 4 – Portfolio models and Asset-Backed Securities

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Objectives of the lecture

Teaching objectives

At the end of this lecture, you will:

- Know how to derive the **loss of a loan portfolio** under a series of assumptions;
- Better understand the **concept of dependence structure** through the theory of **copulas**;
- Be familiar with the **most common credit derivatives** such as CDO and CSO;
- Understand what are these credit derivatives used for and **how they are priced**.

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Vasicek Model – Framework

The Vasicek Model – Purpose and assumptions

Vasicek model's purpose

Vasicek model provides a way to assess **the loss distribution of a portfolio of defaultable assets**.

Assumptions of the infinite homogeneous Vasicek portfolio model

The Vasicek Model usually refers to the infinite homogeneous Vasicek portfolio model that supposes that:

- there is a **countably infinite** number of bonds (loans, mortgages, etc.);
- of **equal nominal value**;
- with the **same maturity**;
- with the **same probability of default** at maturity (PD);
- and with the **same recovery rate** (R).

At the individual bond level, Vasicek Model is a combination of the Merton Model (asset return determines default or non-default) and reduced-form model, with a fixed recovery rate.

Tutorial

The Vasicek Model – Framework

The Vasicek Model – Modeling the returns of the debtors

The Vasicek Model – Definition of the latent variable of return

We define a **latent variable of return**, for each asset as:

$$\forall i \in \mathbb{N}, \qquad R_i = \underbrace{\sqrt{\rho}}_{\substack{\text{correlation}\\ \text{factor}}} \underbrace{F}_{\substack{\text{systemic}\\ \text{factor}}} + \sqrt{1-\rho} \underbrace{e_i}_{\substack{\text{idiosyncratic}\\ \text{factor}}}$$

with $(e_i)_{i\in\mathbb{N}}$ and F are standardized, independent normal variables, and thus $(R_i)_{i\in\mathbb{N}}$ are standardized and correlated, with correlation ρ.

Vasicek Model – Framework

The Vasicek Model – Definition of the default

Definition of the default in the Vasicek model

In the Vasicek model, the bond i **defaults** when:

$$\{R_i < s\}$$

that is when the **latent variable**, R_i, is smaller than s, the **latent threshold** (common for all bonds).

Economic interpretation of the Vasicek model

There is a latent variable for each counterparty in the studied portfolio whose behavior is due to a **(unique) systemic factor** and an **idiosyncratic one**. The latent variable can be understood as some measure of the return of the counterparty, and the systemic factor as a measure of the economic soundness of the economy (GDP, unemployment rate, etc.).

- The smaller F, the harsher the economic environment and the smaller the latent return for **all** the counterparties;
- The smaller e_i, the smaller the return of the **ith** counterparty and the higher its probability of default.

Vasicek Model – Framework

The Vasicek Model – Definition of the default

The Vasicek Model – Link between the latent threshold and the probability of default

We can deduce the expression of the **common latent threshold of default**:
Given that:

$$\text{PD} = \mathbb{P}(R_i < s) = \underbrace{\Phi}_{\substack{\text{Normal}\\ \text{cdf}}}(s)$$

We deduce that:

$$s = \Phi^{-1}(\text{PD})$$

ρ and PD are not outputs of the Vasicek model

ρ and *PD* are input parameters of the model, not outputs. The loss distribution of the portfolio is the output of the model.

Vasicek Model - Loss distribution

What is the distribution of the losses of the portfolio?

The loss distribution of the infinite homogeneous Vasicek portfolio model

We thus have that for **the random variable of the losses of the portfolio**, expressed as a percentage, is:

$$
\begin{aligned}
L \mid F &= \frac{1-R}{N}\sum_{i=1}^{+\infty} \mathbb{1}_{\{R_i < s\}} \\
&= \frac{1-R}{N}\sum_{i=1}^{+\infty} \mathbb{1}_{\left\{e_i < \frac{\Phi^{-1}(PD)-\sqrt{\rho}F}{\sqrt{1-\rho}}\right\}} \\
&\underbrace{=}_{\text{Law of large numbers}} (1-R)\Phi\left(\frac{\Phi^{-1}(PD)-\sqrt{\rho}F}{\sqrt{1-\rho}}\right)
\end{aligned}
$$

Note that L is conditioned to the value of F, the stochastic systemic factor. R is the recovery rate for each loan.

▸ Notebook

Conclusion
Vasicek Model

- Vasicek model assumes an **infinite homogeneous portfolio of correlated loans**;
- The loans are independent conditionally to a **systemic factor**;
- Under assumptions that are **comparable to Merton's approach**, the distribution of the **portfolio loss** can be derived.

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Correlation vs Dependence

Do correlation and dependence refer to the same concept?

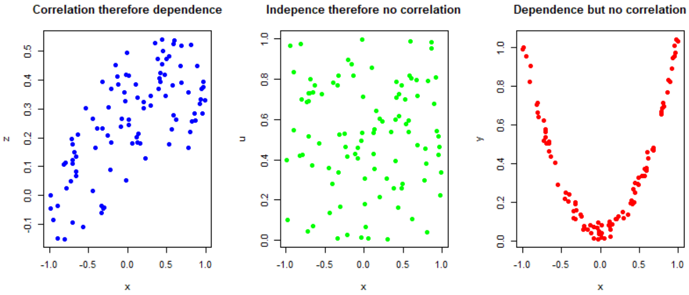
Correlation and dependence

Correlation $\neq$ Dependence

Dependence and correlation **do not refer to the same concept**. In fact, dependence is a much broader concept than correlation. Dependence structures can therefore be much **more complex** than correlation structures.
Analogy: difference between the mean and quantiles.

Correlation vs Dependence

Link between correlation and dependence



Correlation entails dependence but not the other way around!

Copulas - Definition and main results

Definition of a copula

Copula – Definition

A copula is (the cdf of) the joint distribution of random variables $U_1, \ldots, U_d$, each of which being marginally uniformly distributed on $[0;1]$.

$$\forall (u_1, \ldots, u_d) \in [0;1]^d, \quad C(u_1, \ldots, u_d) = \mathbb{P}\left(U_1 \leq u_1, \ldots, U_d \leq u_d\right)$$

It is therefore a function from $[0;1]^d$ to $[0;1]$.

Copulas allow to model the **dependence structure** of a random vector **regardless of its marginal behaviors**.

Copulas - Definition and main results

What is the link between a multivariate distribution and its copula?

Sklar theorem

Sklar's theorem asserts that from any continuous multivariate distribution F, a copula can be deduced with the following formula:

$$\forall (u_1, \ldots, u_d) \in [0;1]^d, \quad C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))$$

and equivalently

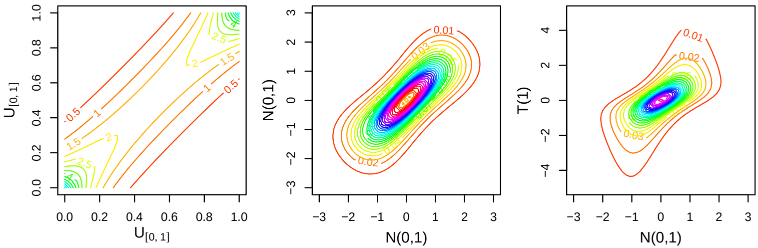
$$\forall (x_1, \ldots, x_d) \in \mathbb{R}^d, \quad F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$$

where $F_1, \ldots, F_d$ are the cdf of the margins of F and $F_1^{-1}, \ldots, F_d^{-1}$ are their inverse distribution functions (or quantile functions).

By transforming the margins of a **continuous** multivariate distribution by their cumulative distribution functions respectively, we obtain the **copula characterizing the dependence structure (and the dependence structure only!)** of this multivariate distribution.

Copulas - Definition and main results

Can we build a multivariate distribution from a copula?



Densities of various bivariate distributions with different marginal distributions but the same dependence structure (Frank copula ($\theta = 7$)).

We can choose **any copula** (to characterize the dependence structure) **and any marginal distributions** (to characterize the marginal behaviors) **to build a multivariate distribution**.

Copulas - Definition and main results

Density function of a copula

We saw that $F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d))$. If F is continuous, by differentiating n times this expression, we can find the joint density, that is:

$$f(x_1, ..., x_d) = f_1(x_1) \times ... \times f_d(x_d) \times \frac{\partial^d C}{\partial x_1 ... \partial x_d}(F_1(x_1), ..., F_d(x_d))$$

With f the density of the joint distribution and $(f_1, ..., f_d)$ the ones of the marginal distributions.

Density of a copula

▸ Definition

We define the **density of a copula**, c:

$$c(u_1, ..., u_d) = \frac{\partial^d C}{\partial u_1 ... \partial u_d}(u_1, ..., u_d) = \frac{f\left(F_1^{-1}(u_1), ..., F_d^{-1}(u_d)\right)}{f_1\left(F_1^{-1}(u_1)\right) \times ... \times f_d\left(F_d^{-1}(u_d)\right)}$$

Estimation of a copula

The multivariate likelihood as a sum of likelihoods

We saw that:

$$f(x_1, ..., x_d) = c(F_1(x_1), ..., F_d(x_d)) \prod_{i=1}^{d} f_i(x_i)$$

where c, is the d-dimensional density of the copula C. In the following, we consider that we have n, d-dimensional observations: $(\mathbf{x_j^{(i)}})$. We can then deduce the likelihood L_C and the loglikelihood LL_C:

$$\begin{aligned} L_C &= \prod_{i=1}^{n} f(x_1^{(i)}, ..., x_d^{(i)}) \\ &= \prod_{i=1}^{n} \left[c(F_1(x_1^{(i)}), ..., F_d(x_d^{(i)})) \prod_{i=j}^{d} f_i(x_j^{(i)}) \right] \\ LL_C &= \sum_{i=1}^{n} \log \left(c(F_1(x_1^{(i)}), ..., F_d(x_d^{(i)})) \right) + \sum_{i=1}^{n} \sum_{i=j}^{d} \log \left(f_i(x_j^{(i)}) \right) \end{aligned}$$

The first term corresponds to **the dependence structure** and the second to **the distributions of the margins**.

Estimation of a copula

How to fit a copula with the Maximum Likelihood Estimator (MLE)?

From now on, we will denote by $\boldsymbol{\theta}$ the parameters of the copula and $\boldsymbol{\alpha}_i$ the parameters of the ith marginal distribution.

They are **two techniques** to fit a copula:

- The Maximum Likelihood Estimator (**MLE**);
- The Inference Functions for Margins method (**IFM**).

The Maximum Likelihood Estimator to fit copulas

The **Maximum Likelihood Estimator** consists in estimating $(\boldsymbol{\theta}, \boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_n)$ by

$$(\boldsymbol{\theta}^{MLE}, \boldsymbol{\alpha}_1^{MLE}, ..., \boldsymbol{\alpha}_n^{MLE})$$

with:

$$(\boldsymbol{\theta}^{MLE}, \boldsymbol{\alpha}_1^{MLE}, ..., \boldsymbol{\alpha}_n^{MLE}) = \operatorname{argmax}_{(\boldsymbol{\theta}, \boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_n)} L((\boldsymbol{\theta}, \boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_n))$$

Estimation of a copula

How to fit a copula with the Inference Functions for Margins (IFM) method?

The Inference Functions for Margins

The **Inference Functions for Margins (IFM)** consists in a two-step procedure:

1. Computing $\forall i \in [1; d], \quad \boldsymbol{\alpha}_{\boldsymbol{i}}^{IFM} = \text{argmax}_{\alpha_i} L_i(\boldsymbol{\alpha}_{\boldsymbol{i}})$
2. Computing $\boldsymbol{\theta}^{IFM} = \text{argmax}_{\boldsymbol{\theta}} L_C(\boldsymbol{\theta}, \hat{\boldsymbol{\alpha}}_{\mathbf{1}}{}^{IFM}, ..., \hat{\boldsymbol{\alpha}}_{\boldsymbol{n}}{}^{IFM})$

Estimation of a copula

Copulas – Difference between MLE and IFM

Difference between MLE and IFM

There is a slight but decisive **difference between the two methods** that confers to both methods different asymptotic properties:

The MLE estimator $(\boldsymbol{\theta}^{MLE}, \boldsymbol{\alpha}_1^{MLE}, ..., \boldsymbol{\alpha}_n^{MLE})$ solves:

$$\left(\frac{\partial L}{\partial \boldsymbol{\theta}}, \frac{\partial L}{\partial \boldsymbol{\alpha}_1}, ..., \frac{\partial L}{\partial \boldsymbol{\alpha}_n} \right) = 0$$

While the IFM one $(\boldsymbol{\theta}^{IFM}, \boldsymbol{\alpha}_1^{IFM}, ..., \boldsymbol{\alpha}_n^{IFM})$ solves:

$$\left(\frac{\partial L}{\partial \boldsymbol{\theta}}, \frac{\partial L_1}{\partial \boldsymbol{\alpha}_1}, ..., \frac{\partial L_n}{\partial \boldsymbol{\alpha}_n} \right) = 0$$

[Joe et al., 1996] shows that MLE and IFM estimation procedures are equivalent in a very particular case: the one where the copula and the margins are Gaussian.

Well-known copulas

The Gaussian copula – Definition

The Gaussian copula

As Gaussian univariate and multivariate cumulative distributions are continuous, applying Sklar's theorem, it can be shown that **the unique Gaussian copula**:

$\forall \mathbf{u} \in [0;1]^d$,

$$
\begin{aligned}
C_{\mathbf{R}}^{\mathcal{N}}(u_1, \ldots, u_d) &= \Phi_{\mathbf{R}}(u_1, \ldots, u_d) \\
&= \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_d)} \frac{1}{(2\pi)^{\frac{d}{2}} \mid \mathbf{R} \mid^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\mathbf{x}^{\star}\mathbf{R}^{-1}\mathbf{x}\right) \mathrm{d}\mathbf{x}
\end{aligned}
$$

Well-known copulas

The Gaussian copula – Density of the copula

Density of the Gaussian copula

Deriving the cdf of the Gaussian copula by each of its component, it can be shown that **the density of the Gaussian copula** with correlation matrix $\mathbf{R}$:

$$\forall \mathbf{u} \in [0;1]^d, \quad c_{\mathbf{R}}^{\mathcal{N}}(u_1, \ldots, u_d) = \frac{1}{\sqrt{|\mathbf{R}|}} \exp\left(-\frac{1}{2} \begin{pmatrix} \Phi^{-1}(u_1) \\ \vdots \\ \Phi^{-1}(u_d) \end{pmatrix}^{\star} \cdot \left(\mathbf{R}^{-1} - \mathbf{I}_d\right) \cdot \begin{pmatrix} \Phi^{-1}(u_1) \\ \vdots \\ \Phi^{-1}(u_d) \end{pmatrix} \right)$$

Well-known copulas

The Gaussian copula – Simulation

It often happens that modeling involves complex univariate and multivariate variables so that there is no close formula to compute the risk metric: in such a case, one must use Monte Carlo techniques and thus simulate the copula.

How to simulate a Gaussian copula?

In order to **simulate a Gaussian copula** $C_{\mathbf{R}}^{\mathcal{N}}$, one must apply this two-step procedure:

1. First, one must **simulate a normal reduced centered vector** with correlation matrix R, $\boldsymbol{X} = (X_1, X_2, ..., X_d)$;
2. Second, one must **compose each variable of the vector by the inverse cumulative distribution function of a univariate centered and reduced Gaussian distribution**, $(\Phi(X_1), ..., \Phi(X_d))$.

And it follows the same procedure for any other copula deduced from a multivariate distribution applying Sklar's theorem.

Well-known copulas

Other well-known copulas

Other well-known copulas

There are other well-known copulas:

- **Other copulas deduced from multivariate distributions** applying Sklar's theorem: Student copulas, grouped t-copulas, individual t-copulas, etc.;
- **the so-called Archimedean copulas**, that can be written as:

$$C(u_1, \ldots, u_d; \theta) = \psi^{-1}\left(\psi(u_1; \theta) + \cdots + \psi(u_d; \theta); \theta\right)$$

where $\psi\colon [0,1] \times \Theta \to [0, \infty)$ is a continuous, strictly decreasing and convex function such that $\psi(1; \theta) = 0$, called the **generator of the Archimedean copula**.

Tutorial | Notebook | Quiz

Portfolio models and copulas

Link between the Vasicek model and copulas

Vasicek model and Gaussian copula

The Vasicek model is a copula-based model. Indeed, the dependence structure between the default times is based on a Gaussian copula.

The formalization of such a point was made in [Burtschell et al., 2008].

▸ Notebook

Extension of the Vasicek model based on other copulas

Many models can be deduced from this finding:

- for a more extreme dependence structure, one can use a **Student copula** to link default times;
- for an asymmetric dependence structure of the default times, one can use the **Gumbel copula**;
- etc.

Conclusion

Copulas

- **Correlation** is only **one aspect of the concept of dependence**;
- Copulas are a tool to model **complex structures of dependence**;
- Copulas allow to **capture dependence structures separately from the marginal behavior**;
- The **estimation** of copulas remains a **complex issue**;
- Going **beyond the Gaussian** copulas is crucial to **model the reality more accurately**.

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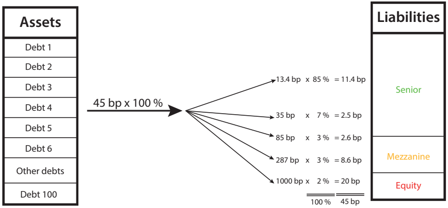
Collateralized Debt Obligation (CDO)

CDO capital structure

Collateralized Debt Obligation – Capital structure

- A **SPV (Special Vehicle Purpose)** issues several tranches of debts to buy assets (debt instruments);
- The **tranches are rated** by rating agencies (Fitch, Moody's, S&P);
- The **tranches offer different risk / return ratios**:
  - Losses impact first the junior tranches;
  - Principal cash-flows are redirected to the more senior tranches first.

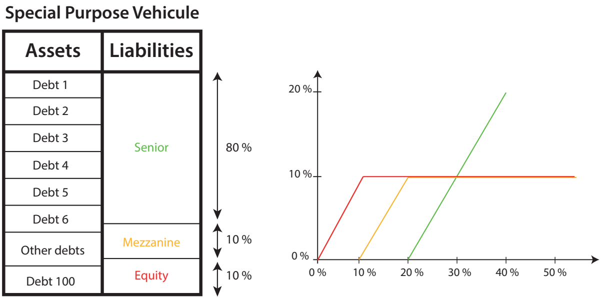


Collateralized Debt Obligation (CDO)

Option theory and description of CDO

Let L be the percentage of losses:

- If L is smaller than 10%: losses only affect equity;
- If L is between 10% and 20% : losses affect equity and mezzanine;
- If L is larger than 20%: losses affect all the tranches.



Therefore, tranching is inherently **non-linear** operation.

Collateralized Debt Obligation (CDO)

CDO's economic purposes (I/III)

Balance sheet CDO	Arbitrage CDO
▶ Refinancing of a portfolio (private investors, ECB, etc); ▶ A bank wants to transfer the risk of its loan portfolio; ▶ Balance-sheet reduction; ▶ Regulatory and economic capital optimization; ▶ Increase ROE and RAROC; ▶ Close a business line.	▶ An asset manager wants to build a corporate portfolio; ▶ Funding through the issuance of debt securities and equity; ▶ That generates management and structuration fees; ▶ Increases Assets under Management (AuM); ▶ And offers diversification to the clients.

These are impacted by the EU's Simple, Transparent and Standardised (STS) Securitisations.

Collateralized Debt Obligation (CDO)

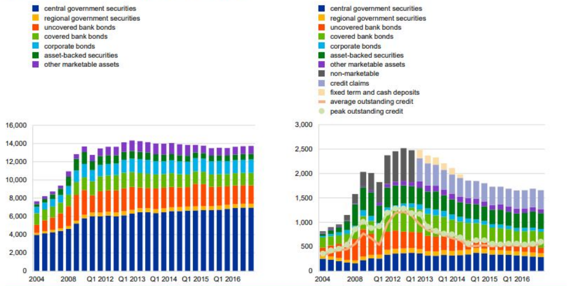
CDO's economic purposes (III/III)

CDOs/Securitized products provide refinancing opportunities.

Figure 3

Eligible assets and use of collateral

(EUR billions; left-hand side: eligible assets; right-hand side: use of collateral)



Source: ECB.
Notes: collateral used is reported after valuation and haircuts in averages of end of month data over each time period shown.
Since 2013 Q1, the category "Non-marketable assets" is split into two categories: "Fixed term and cash deposits" and "Credit claims".
Last observation: 2016 Q4.

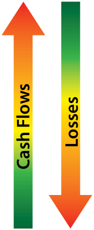
Collateralized Debt Obligation (CDO)

CDOs economic purposes (III/III)

CDO intends to offer the optimal spread/rating duo for every investor.

Special Purpose Vehicule

Assets	Liabilities
Debt 1	
Debt 2	
Debt 3	Tranche AAA
Debt 4	
Debt 5	
Debt 6	
Other debts	Tranche A
Debt 100	Equity



- **The senior tranch is generally rated AAA;**
- **One or several mezzanine tranches are rated AAA to B;**
- **The equity tranche is generally not rated.**

For more details on the subject, you can take a look at [Bluhm and Christian, 2003].

Tutorial

Collateralized Debt Obligation (CDO)

The concept of credit enhancement

Credit enhancement

There are several ways to improve the credit profile of an ABS:

- **Excess spread**: the received rate is higher than the served one;
- **Overcollateralization**: the face value of the underlying loan portfolio is larger than the security it backs;
- **Monolines and wrapped securities**: CDS on the underlying assets are bought from monolines to cover part of the losses.

Collateralized Debt Obligation (CDO)

Pricing of CDO (I/III)

Expected loss on tranche $[A; D]$

The **expected loss** at time t on tranche $[A; D]$, EL_t, is a simple function of the loss on the underlying portfolio at time t:

$$\mathrm{EL}_t = \mathbb{E}((L(t) - A)^+ - (L(t) - D)^+)$$

The loss distribution function

For a granular homogeneous credit portfolio, the loss at time t depends on the systemic factor F and the default time cdf H at time t, and **the loss distribution function** expression is:

$$L(t, F) = (1 - R)\Phi\left(\frac{\Phi^{-1}(\mathrm{H(t)}) - \sqrt{\rho}F}{\sqrt{1-\rho}}\right)$$

Collateralized Debt Obligation (CDO)

Pricing of CDO (II/III)

Floating leg market value

The **floating leg market value** of the CDO tranche $[A; D]$ is:

$$\mathrm{JV}^{[A;D]}(0; T) = \int_0^T e^{-rt} \mathrm{dEL}_t = e^{-rT}\mathrm{EL}_T + r\int_0^T e^{-rt}\mathrm{EL}_t \mathrm{d}t$$

Fix leg market value

The **fixed leg market value** of the CDO tranche $[A; D]$ is:

$$\begin{aligned}
\mathrm{JF}^{[A;D]}(0; T) &= s^{[A;D]} \int_0^T e^{-rt}(D - A - \mathrm{EL}_t)\mathrm{d}t \\
&= s^{[A;D]}\mathrm{DV}^{[A;D]}(0; T) \\
&= s^{[A;D]} \left(\left(\frac{D - A}{r} \right) (1 - e^{-rT}) - \frac{1}{r}(\mathrm{JV}^{[A;D]}(0; T) - e^{-rT}\mathrm{EL}_T \right)
\end{aligned}$$

Collateralized Debt Obligation (CDO)

Pricing of CDO (III/III)

As for CDS, we can use the no arbitrage assumption to calculate the spread of the studied CDO tranche.

Spread of the tranche of a CDO

Thus, the **spread of the CDO tranche** $[A; D]$ is:

$$s^{[A;D]} = \frac{JF^{[A;D]}(0, T)}{DV^{[A;D]}(0, T)}$$

▸ Notebook

Collateralized Synthetic Obligation (CSO)

CSO vs CDO

Cash	**Synthetic**
Large AAA size	Mezzanine AAA+ large super senior
High funding cost	Low funding cost
Limited invested universe	Very large investment universe
Transfer of the assets	Risk transfer only
High management fees	Low management fees
10-15 % high yield	100% investment grade
Average rating BBB-	Average rating A
Low leverage (equity 10 %)	High leverage (2 / 3 %)

Collateralized Synthetic Obligation (CSO)

CSO and CDS indices

Credit Index

iTraxx is a Credit Index used in Europe and Asia with 125 references (the equivalent in the US is **CDX**). It has the following characteristics:

- Spreads are usually from 10 bp to 120 bp with an average around 35 bp;
- Spread volatility is around 2 bp a day;
- Listed tranches are $[0\%, 3\%]$, $[3\%, 6\%]$, $[6\%, 9\%]$, $[9\%, 12\%]$, $[12\%, 22\%]$;
- Maturities are of 3, 5, 7, 10 years, rolled every 6 months;

▸ iTraxx indice

Implied correlation and base correlation

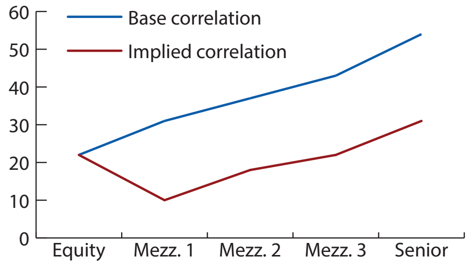
Implied correlation and base correlation

Implied correlation of tranche $[A; D]$

The **implied correlation** of $[A; D]$, knowing the spread of the tranche $s_{A,D}$, is the correlation required in the Vasicek model to price the CSO of tranche $[A; D]$, $s^{[A,D]}$.

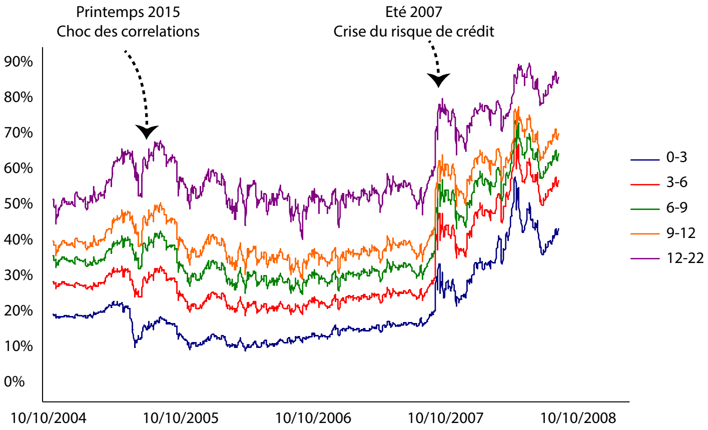
Base correlation

The **base correlation** in K is the implied correlation of $[0; K]$.



Implied correlation and base correlation

Base correlations dynamics



Implied correlation and base correlation

Implied correlation and base correlation – Bijectivity with CDO tranche prices

Bijective relationship between base correlation and CDO tranches spreads

There is a **bijective relationship** between the base correlation and the spread of a CDO tranche :

$$s^{[A;D]} = \frac{JV^{[0;D]}(\rho^{[0;D]}) - JV^{[0;A]}(\rho^{[0;A]})}{DV^{[0;D]}(\rho^{[0;D]}) - DV^{[0;A]}(\rho^{[0;A]})}$$

As option traders usually quote prices with implicit volatilities, CDO traders quote their prices using base correlations.

Implied correlation and base correlation

Implied correlation and base correlation – Interpretation

What do implied and base correlations tell us?

[D'Amato et al., 2005] presents several possible explanations for the correlation smile:

- there is a **segmentation among investors** across tranches;
- the used **models are inefficient**.

Implied correlation and base correlation

Implied correlation and base correlation – Limits of the Vasicek model to price CDO tranches

Wall Street's Math Wizards Forgot a Few Variables

By STEVE LOHR SEPT. 12, 2009

IN the aftermath of the great meltdown of 2008, Wall Street's quants have been cast as the financial engineers of profit-driven innovation run amok. They, after all, invented the exotic securities that proved so troublesome.

▸ The New York Times

Were models the reason for the subprime crisis?

The method used to price CDO tranches has been proven wrong:

- they are too many **homogeneity assumptions** (for correlation, default, maturity, nominal, etc.);
- the **dependence structure** in the model is not extreme enough.

There are a lot of other reasons (quality of the data – Garbage In Garbage Out logic among others) why the subprime crisis happened, most of them will be presented during the Subprimes Crisis Case Study (Lecture 7).

Delta hedging

Delta hedging

A trader wants to buy a protection on a mezzanine tranche;

- He hedges the market value fluctuations of his book by selling protection on individual CDS names;
- Trader's book value is:

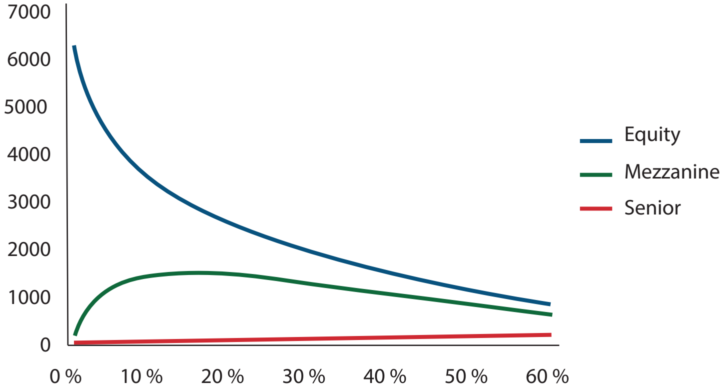
$$P(t) = V_{Tr}(t) + \sum_i \Delta_i V_{CDS_i}(t)$$

Thus, the **hedge ratio** is:

$$\frac{\partial P(t)}{\partial s_j} = 0 \Rightarrow \Delta_j = \frac{\partial V_{Tr}(t)}{DV_j \partial s_j}$$

Hedging single tranche exposure

Pricing sensitivy to correlation



Conclusion
CDO and CSO

- **Collateralized Debt Obligations (CDOs) offer varying levels of risk and return** through structured tranches, meeting investor demands for yield across the credit spectrum;
- **Collateralized Synthetic Obligations (CSOs)** offer similar risk structures but derive from **synthetic assets** like credit default swaps (CDS) rather than actual bonds, allowing for greater flexibility and reduced funding costs;
- The **complexity of pricing these products**, particularly given the flaws in correlation models, contributed to financial instability during the subprime crisis.

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First-To-Default products – Definition

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First-To-Default product

FtD products are similar to CDS contracts except that:

- They are based on a pool of **10 names maximum**;
- The protection buyer pays a **constant spread up to the first default** on the reference basket (if it occurs before maturity);
- When (and if) the first default occurs the protection buyer **delivers the defaulted bond and receives par**.

If the underlying assets were perfectly dependent, the FtD would be equivalent to a single-name CDS.

First-To-Default purpose and arbitrage bounds

First-To-Default purpose and arbitrage bounds

First-To-Default purpose

- The FtD is **riskier than the most risky reference** entity of the basket;
- Buying FtD protection is **cheaper** than buying the protection of each reference name in the basket.

Arbitrage bound of FtD products

Let $(s_1, ..., s_d)$ be the spreads of the underlying names, we have that **the spread of the FtD, s_{FtD} arbitrage bounds** are:

$$\max(s_1, ..., s_d) \leq s_{FtD} \leq \sum_{i=1}^{d} s_i$$

It is a consequence of the no arbitrage assumption.

Rule of thumb for FtD pricing

$$s_{FtD} \approx \frac{2}{3} \sum_{i=1}^{d} s_i$$

Other synthetic products

Other synthetic products (I/II)

Other synthetic products

- **CDO squared**
  - Synthetic CDO on mezzanine synthetic single tranches;
  - More leverage;
  - Caution to systemic risk and overlaps.
- **Leveraged super senior**
  - Super senior tranche leveraged 6-10 times;
  - AAA rating, spread = 60 pb instead of 15 pb;
  - More credit enhancement compared to mezzanine AAA.
- **Combo notes**
  - Combination of A mezzanine and equity;
  - Principal rated A- by rating agencies.

Other synthetic products

Other synthetic products (II/II)

Other synthetic products

- **EDS: Equity Default Swap**
  - An "equity event" replaces the usual "credit event";
  - The floating leg of the swap pays a cash-flow when the underlying stock hits the threshold of 30% of its value at inception;
  - Require an equity-credit model.
- **CEO: Collateralized Equity Obligation**
  - For example a CDO of EDS or private equity;
  - In some cases, the maturity of the assets is an issue (ex: private equity).
- **CFO: Collateralized Fund Obligation**
  - CDO collateralized by shares of mutual funds or hedge funds.

▸ Quiz

Conclusion

Other credit products

- **Other synthetic credit products** such as FtD, EDS, and CFOs represent the growing complexity and innovation in the credit derivatives market, offering **unique ways to manage and transfer credit risk**;
- These instruments allow investors to target specific risk exposures, often with enhanced leverage, but also come with **increased risk and correlation challenges**.

Conclusion

Portfolio models and Asset-Backed Securities

- The **Vasicek model and its extensions remain the predominant frameworks** for assessing credit risk in portfolios, offering a foundation for understanding default probabilities and credit spread dynamics;
- Properly modeling correlation is critical in credit risk assessment - by **moving beyond linear and Gaussian assumptions, tools like copulas allow for a more accurate representation** of complex dependencies between defaults in a portfolio;
- Accurate credit risk modeling facilitates the **design of sophisticated credit products**, both funded and synthetic, such as CLOs and CSOs, which cater to varying investor risk appetites and yield preferences;
- Despite their potential, these products present challenges due to the **inherent difficulty in accurately modeling correlations**, making risk management crucial to avoid mispricing and systemic risk.

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