Credit Risk

Lecture 4 – Portfolio models and Asset Back Securities

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Objectives of the lecture

Teaching objectives

At the end of this lecture, you will:

- Know how to derive the **loss of a loan portfolio** under a series of assumptions;
- Better understand the **concept of dependence structure** through the theory of **copulas**;
- Be familiar with the **most common credit derivatives** such as CDO and CSO;
- Understand what are these credit derivatives used for and **how they are priced**.

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Vasicek Model – Framework

The Vasicek Model – Purpose and assumptions

Vasicek model's purpose

Vasicek model provides a way to assess **the loss distribution of a portfolio of defaultable assets**.

Assumptions of the infinite homogeneous Vasicek portfolio model

The Vasicek Model usually refers to the infinite homogeneous Vasicek portfolio model that supposes that:

- there is a **countable infinite** number of bonds (loans, mortgages, etc.);
- of **equal nominal**;
- **same maturity**;
- **same probability of default** at maturity (PD);
- and with the **same recovery rate** (R).

At individual bond level, Vasicek Model is a combination of Merton Model (asset return determines default or non-default) and reduced-form model, with a fixed recovery rate.

Tutorial

The Vasicek Model – Framework

The Vasicek Model – Modeling the returns of the debtors

The Vasicek Model – Definition of the latent variable of return

We define a **latent variable of return**, for each asset as:

$$\forall i \in \mathbb{N}, \qquad R_i = \underbrace{\sqrt{\rho}}_{\text{corelation factor}} \underbrace{F}_{\text{systemic factor}} + \sqrt{1-\rho} \underbrace{e_i}_{\text{idiosyncratic factor}}$$

with $(e_i)_{i\in\mathbb{N}}$ and F are standardized, independent, normal variables, and thus $(R_i)_{i\in\mathbb{N}}$ are standardized and correlated, with correlation ρ.

Vasicek Model – Framework

The Vasicek Model – Definition of the default

Definition of the default in the Vasicek model

In the Vasicek model, the bond i **defaults** when:

$$\{R_i < s\}$$

that is when the **latent variable**, R_i, is smaller than s, the **latent threshold** (common for all bonds).

Economic interpretation of the Vasicek model

There is a latent variable for each counterparty in the studied portfolio whose behavior is due to a **(unique) systemic factor** and an **idiosyncratic one**. The latent variable can be understood as some measure of the return of the counterparty, and the systemic factor as a measure of the economic soundness of the economy (GDP, unemployment rate, etc.).

- The smaller F, the harsher the economic environment and the smaller the latent return for **all** the counterparties;
- The smaller e_i, the smaller the return of the **ith** counterparty and the higher its probability of default.

Vasicek Model – Framework

The Vasicek Model – Definition of the default

The Vasicek Model – Link between the latent threshold and the probability of default

We can deduce the expression of the **common latent threshold of default**:
Given that:

$$\mathrm{PD} = \mathbb{P}(R_i < s) = \underbrace{\Phi}_{\text{Normal cdf}}(s)$$

We deduce that:

$$s = \Phi^{-1}(\mathrm{PD})$$

ρ and PD are not outputs of the Vasicek model

ρ and *PD* are input parameters of the model, not outputs. The loss distribution of the portfolio is the output of the model.

Vasicek Model - Loss distribution

What is the distribution of the losses of the portofolio?

The loss distribution of the infinite homogeneous Vasicek portfolio model

We thus have that for **the random variable of the losses of the portfolio**, expressed as a percentage, is:

$$
\begin{aligned}
L \mid F &= \frac{1-R}{N} \sum_{i=1}^{+\infty} \mathbb{1}_{\{R_i < s\}} \\
&= \frac{1-R}{N} \sum_{i=1}^{+\infty} \mathbb{1}_{\left\{e_i < \frac{\Phi^{-1}(PD) - \sqrt{\rho} F}{\sqrt{1-\rho}}\right\}} \\
&\underbrace{=}_{\text{Law of large numbers}} (1-R)\Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho} F}{\sqrt{1-\rho}}\right)
\end{aligned}
$$

Note that L is conditioned to the value of F, the stochastic systemic factor. R is the recovery rate for each loan.

▸ Notebook

Conclusion

Vasicek Model

- Vasicek model assumes an **infinite homogeneous portfolio of correlated loans**;
- The loans are independent conditionnally to a **systemic factor**;
- Under assumptions **comparable to Merton's approach**, the distribution of the **portfolio loss** can be derived.

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Correlation vs Dependence

Do correlation and dependence refer to the same concept?

Correlation and dependence

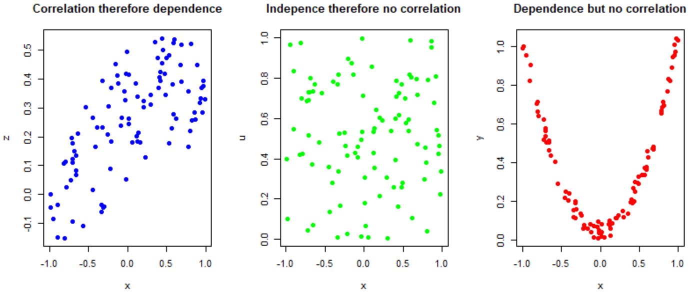
Correlation $\neq$ Dependence

Dependence and correlation **do not refer to the same concept**. In fact, dependence is a much broader concept than correlation. Dependence structures can therefore be much **more complex** than correlation structures.

Analogy: difference between mean and quantiles.

Correlation vs Dependence

Link between correlation and dependence



Correlation entails dependence but not the other way around!

Copulas - Definition and main results

Definition of a copula

Copula – Definition

A copula is (the cdf of) the joint distribution of random variables $U_1, \ldots, U_d$, each of which being marginally uniformly distributed on $[0;1]$.

$$\forall (u_1, \ldots, u_d) \in [0;1]^d, \quad C(u_1, \ldots, u_d) = \mathbb{P}\left(U_1 \leq u_1, \ldots, U_d \leq u_d\right)$$

It is therefore a function from $[0;1]^d$ to $[0;1]$.

Copulas allow to model the **dependence structure** of a random vector **regardless of its marginal behaviors**.

Copulas - Definition and main results

What is the link between a multivariate distribution and its copula?

Sklar theorem

Sklar's theorem asserts that from any continuous multivariate distribution F, a copula can be deduced with the following formula:

$$\forall (u_1, \dots, u_d) \in [0;1]^d, \quad C(u_1, \dots, u_d) = F(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d))$$

and equivalently

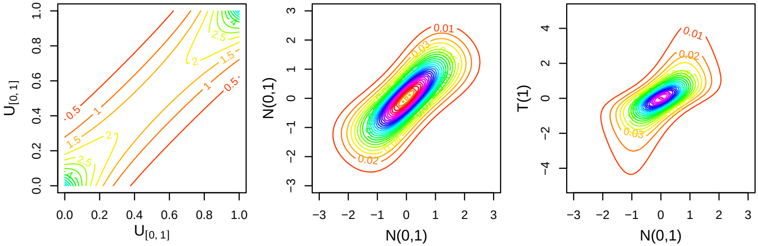
$$\forall (x_1, \dots, x_d) \in \mathbb{R}^d, \quad F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d))$$

where $F_1, \dots, F_d$ are the cdf of the margins of F and $F_1^{-1}, \dots, F_d^{-1}$ are their inverse distribution functions (or quantile functions).

By transforming the margins of a **continuous** multivariate distribution by their cumulative distribution functions respectively, we obtain the **copula characterizing the dependence structure (and the dependence structure only!)** of this multivariate distribution.

Copulas - Definition and main results

Can we build a multivariate distribution from a copula?



Densities of various bivariate distributions with different marginal distributions but the same dependence structure (Frank copula ($\theta = 7$)).

We can choose **any copula** (to characterize the dependence structure) **and any marginal distributions** (to characterize the marginal behaviors) **to build a multivariate distribution**.

Copulas - Definition and main results

Density function of a copula

We saw that $F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d))$. If F is continuous, by derivating n times this expression, we can find the joint density, that is:

$$f(x_1, ..., x_d) = f_1(x_1) \times ... \times f_d(x_d) \times \frac{\partial^d C}{\partial x_1 ... \partial x_d}(F_1(x_1), ..., F_d(x_d))$$

With f the density of the joint distribution and $(f_1, ..., f_d)$ the ones of the marginal distributions.

Density of a copula

Definition

We define the **density of a copula**, c:

$$c(u_1, ..., u_d) = \frac{\partial^d C}{\partial u_1 ... \partial u_d}(u_1, ..., u_d) = \frac{f\left(F_1^{-1}(u_1), ..., F_d^{-1}(u_d)\right)}{f_1\left(F_1^{-1}(u_1)\right) \times ... \times f_d\left(F_d^{-1}(u_d)\right)}$$

Estimation of a copula

The multivariate likelihood as a sum of likelihoods

We saw that:

$$f(x_1, ..., x_d) = c(F_1(x_1), ..., F_d(x_d)) \prod_{i=1}^{d} f_i(x_i)$$

where c, is the d-dimensional density of the copula C. In the following, we consider that we have n, d-dimensional observations: $(\mathbf{x_j^{(i)}})$. We can then deduce the likelihood L_C and the loglikelihood LL_C:

$$\begin{aligned} L_C &= \prod_{i=1}^{n} f(x_1^{(i)}, ..., x_d^{(i)}) \\ &= \prod_{i=1}^{n} \left[c(F_1(x_1^{(i)}), ..., F_d(x_d^{(i)})) \prod_{i=j}^{d} f_i(x_j^{(i)}) \right] \\ LL_C &= \sum_{i=1}^{n} \log \left(c(F_1(x_1^{(i)}), ..., F_d(x_d^{(i)})) \right) + \sum_{i=1}^{n} \sum_{i=j}^{d} \log \left(f_i(x_j^{(i)}) \right) \end{aligned}$$

The first term corresponds to **the dependence structure** and the second to **the distributions of the margins**.

Estimation of a copula

How to fit a copula with the Maximum Likelihood Estimator (MLE)?

From now on, we will denote by $\boldsymbol{\theta}$ the parameters of the copula and $\boldsymbol{\alpha}_i$ the parameters of the ith marginal distribution.

They are **two techniques** to fit a copula:

- The Maximum Likelihood Estimator (**MLE**);
- The Inference Functions for Margins method (**IFM**).

The Maximum Likelihood Estimator to fit copulas

The **Maximum Likelihood Estimator** consists in estimating $(\boldsymbol{\theta}, \boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_n)$ by

$$(\boldsymbol{\theta}^{MLE}, \boldsymbol{\alpha}_1^{MLE}, ..., \boldsymbol{\alpha}_n^{MLE})$$

with:

$$(\boldsymbol{\theta}^{MLE}, \boldsymbol{\alpha}_1^{MLE}, ..., \boldsymbol{\alpha}_n^{MLE}) = \text{argmax}_{(\boldsymbol{\theta}, \boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_n)} L((\boldsymbol{\theta}, \boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_n))$$

Estimation of a copula

How to fit a copula with the Inference Functions for Margins (IFM) method?

The Inference Functions for Margins

The **Inference Functions for Margins (IFM)** consists in a two-step procedure:

1. Computing $\forall i \in [1; d], \quad \boldsymbol{\alpha}_{\boldsymbol{i}}^{IMF} = \text{argmax}_{\alpha_i} L_i(\boldsymbol{\alpha}_{\boldsymbol{i}})$
2. Computing $\boldsymbol{\theta}^{IMF} = \text{argmax}_{\boldsymbol{\theta}} L_C(\boldsymbol{\theta}, \hat{\boldsymbol{\alpha}}_{\mathbf{1}}{}^{IFM}, \ldots, \hat{\boldsymbol{\alpha}}_{\boldsymbol{n}}{}^{IFM})$

Estimation of a copula

Copulas – Difference between MLE and IFM

Difference between MLE and IFM

There is a slight but decisive **difference between the two methods** that confers to both methods different asymptotic properties:

The MLE estimator $(\boldsymbol{\theta}^{MLE}, \boldsymbol{\alpha}_{\mathbf{1}}^{MLE}, ..., \boldsymbol{\alpha}_{\boldsymbol{n}}^{MLE})$ solves:

$$\left(\frac{\partial L}{\partial \boldsymbol{\theta}}, \frac{\partial L}{\partial \boldsymbol{\alpha}_{\mathbf{1}}}, ..., \frac{\partial L}{\partial \boldsymbol{\alpha}_{\boldsymbol{n}}}\right) = 0$$

While the IFM one $(\boldsymbol{\theta}^{IFM}, \boldsymbol{\alpha}_{\mathbf{1}}^{IFM}, ..., \boldsymbol{\alpha}_{\boldsymbol{n}}^{IMF})$ solves:

$$\left(\frac{\partial L}{\partial \boldsymbol{\theta}}, \frac{\partial L_1}{\partial \boldsymbol{\alpha}_{\mathbf{1}}}, ..., \frac{\partial L_n}{\partial \boldsymbol{\alpha}_{\boldsymbol{n}}}\right) = 0$$

[Joe et al., 1996] shows that MLE and IFM estimation procedures are equivalent in a very particular case: the one where the copula and the margins are Gaussian.

Well-know copulas

The Gaussian copula – Definition

The Gaussian copula

As Gaussian univariate and multivariate cumulative distributions are continuous, applying Sklar's theorem, it can be shown that **the unique Gaussian copula**:

$\forall \mathbf{u} \in [0;1]^d$,

$$
\begin{aligned}
C_{\mathbf{R}}^{\mathcal{N}}(u_1, \ldots, u_d) &= \Phi_{\mathbf{R}}(u_1, \ldots, u_d) \\
&= \int_{-\infty}^{\Phi^{-1}(u_1)} \ldots \int_{-\infty}^{\Phi^{-1}(u_d)} \frac{1}{(2\pi)^{\frac{d}{2}} \mid \mathbf{R} \mid^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\mathbf{x}^{\star}\mathbf{R}^{-1}\mathbf{x}\right) \mathrm{d}\mathbf{x}
\end{aligned}
$$

Well-know copulas

The Gaussian copula – Density of the copula

Density of the Gaussian copula

Deriving the cdf of the Gaussian copula by each of its component, it can be shown that **the density of the Gaussian copula** with correlation matrix $\mathbf{R}$:

$$\forall \mathbf{u} \in [0;1]^d, \quad c_{\mathbf{R}}^{\mathcal{N}}(u_1, \ldots, u_d) = \frac{1}{\sqrt{|\mathbf{R}|}} \exp\left(-\frac{1}{2} \begin{pmatrix} \Phi^{-1}(u_1) \\ \vdots \\ \Phi^{-1}(u_d) \end{pmatrix}^{\star} \cdot \left(\mathbf{R}^{-1} - \mathbf{I}_d\right) \cdot \begin{pmatrix} \Phi^{-1}(u_1) \\ \vdots \\ \Phi^{-1}(u_d) \end{pmatrix} \right)$$

Well-know copulas

The Gaussian copula – Simulation

It often happens that modeling involves complex univariate and multivariate variables so that there is no close formula to compute the risk metric: in such a case, one must use Monte Carlo techniques and thus simulate the copula.

How to simulate a Gaussian copula?

In order to **simulate a Gaussian copula** $C_{\mathbf{R}}^{\mathcal{N}}$, one must apply this two-step procedure:

1. First, one must **simulate a normal reduced centered vector** with correlation matrix R, $\boldsymbol{X} = (X_1, X_2, ..., X_d)$;
2. Second, one must **compose each variable of the vector by the inverse cumulative distribution function of a univariate centered and reduced Gaussian distribution**, $(\Phi(X_1), ..., \Phi(X_d))$.

And its goes the same way for any other copula deduced from a multivariate distribution applying Sklar's theorem.

Well-know copulas

Other well-known copulas

Other well-known copulas

There are other well-known copulas:

- **Other copulas deduced from multivariate distributions** applying Sklar's theorem: Student copulas, grouped t-copulas, individual t-copulas, etc.;
- **the so-called Archimedean copulas**, that can be written as:

$$C(u_1, \ldots, u_d; \theta) = \psi^{-1}(\psi(u_1; \theta) + \cdots + \psi(u_d; \theta); \theta)$$

where $\psi : [0,1] \times \Theta \to [0, \infty)$ is a continuous, strictly decreasing and convex function such that $\psi(1; \theta) = 0$, called the **generator of the Archimedean copula**.

Tutorial | Notebook | Quiz

Portfolio models and copulas

Link between the Vasicek model and copulas

Vasicek model and Gaussian copula

The Vasicek model is a copula-based model. Indeed, the dependence structure between the default times is based on a Gaussian copula.

> The formalization of such a point was made in [Burtschell et al., 2008].

▸ Notebook

Extension of the Vasicek model based on other copulas

A lot of models can be deduced from this finding:

- for a more extreme dependence structure, one can use a **Student copula** to link default times;
- for an assymetric dependence structure of the default times, one can use the **Gumbel copula**;
- etc.

Conclusion

Copulas

- **Correlation** is only **one aspect of the concept of dependence**;
- Copulas are a tool to model **complex structures of dependence**;
- Copulas allow to **apprehend dependence structures separately from the marginal behavior**;
- The **estimation** of copulas remains a **complex issue**;
- Going **beyond the Gaussian** copulas is crucial to **model the reality more accurately**.

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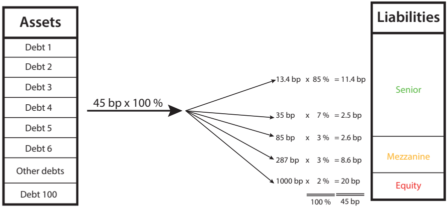
Collateralized Debt Obligation (CDO)

CDO capital structure

Collateralized Debt Obligation – Capital structure

- A **SPV (Special Vehicule Purpose)** issues several tranches of debts to buy assets (debt instruments);
- The **tranches are rated** by rating agencies (Fitch, Moody's, S&P);
- The **tranches offer different risk / return ratios**:
  - Losses impact first the junior tranches;
  - Principal cash-flows are redirected to senior tranche first.

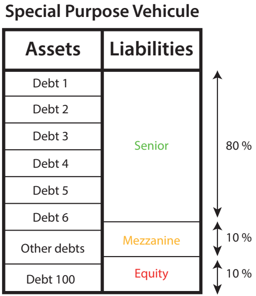


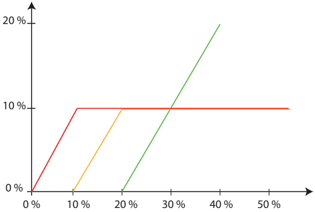
Collateralized Debt Obligation (CDO)

Option theory and description of CDO

Let L be the percentage of losses:

- If L is smaller than 10%: losses only affect equity;
- If L is between 10% and 20% : losses affect equity and mezzanine;
- If L is larger than 20%: losses affect all the tranches.





So tranching is a **non-linear** operation.

Collateralized Debt Obligation (CDO)

CDO's economic purposes (I/III)

Balance sheet CDO	Arbitrage CDO
▶ Refinancing of a portfolio (Private investors, ECB, etc); ▶ A bank wants to transfer the risk of its loan portfolio; ▶ Balance-sheet reduction; ▶ Regulatory and economic capital optimization; ▶ Increase ROE and RAROC; ▶ Close a business line.	▶ An asset manager wants to build a corporate portfolio; ▶ Funding through the issuance of debt securities and equity; ▶ That generates management and structuration fees; ▶ Increases Asset under Management (AuM); ▶ And offers diversification to the clients.

These are impacted by the EU's Simple, Transparent and Standardised (STS) Securitisations.

Collateralized Debt Obligation (CDO)

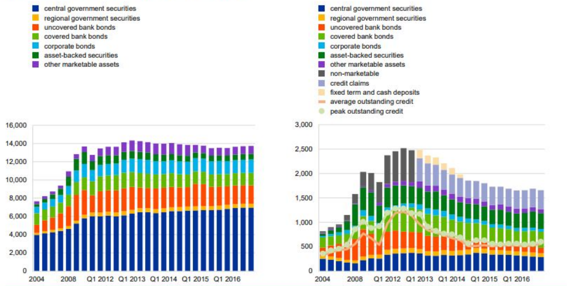
CDO's economic purposes (III/III)

CDOs/Securitized products offer refinancing possibilities.

Figure 3

Eligible assets and use of collateral

(EUR billions; left-hand side: eligible assets; right-hand side: use of collateral)



Source: ECB.
Notes: collateral used is reported after valuation and haircuts in averages of end of month data over each time period shown.
Since 2013 Q1, the category "Non-marketable assets" is split into two categories: "Fixed term and cash deposits" and "Credit claims".
Last observation: 2016 Q4.

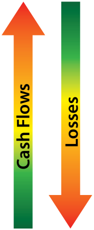
Collateralized Debt Obligation (CDO)

CDO's economic purposes (III/III)

CDO intends to offer the optimal spread/rating duo for every investor.

Special Purpose Vehicule

Assets	Liabilities
Debt 1	Tranche AAA
Debt 2	
Debt 3	
Debt 4	
Debt 5	
Debt 6	
Other debts	Tranche A
Debt 100	Equity



- The senior tranch is generally rated AAA;
- One or several mezzanine tranches are rated AAA to B;
- The equity tranche is generally not rated.

For more details on the subject, you can take a look at [Bluhm and Christian, 2003].

Tutorial

Collateralized Debt Obligation (CDO)

The concept of credit enhancement

Credit enhancement

There are several way to improve the credit profile of an ABS:

- **Excess spread**: the received rate is higher than the served one;
- **Overcollateralization**: the face value of the underlying loan portfolio is larger than the security it backs;
- **Monolines and wrapped securities**: CDS on the underlying assets are bought to monolines to cover part of the losses.

Collateralized Debt Obligation (CDO)

Pricing of CDO (I/III)

Expected loss on tranche $[A; D]$

The **expected loss** at time t on tranche $[A; D]$, EL_t, is a simple function of the loss on the underlying portfolio at time t:

$$\text{EL}_t = \mathbb{E}((L(t) - A)^+ - (L(t) - D)^+)$$

The loss distribution function

For a granular homogeneous credit portfolio, the loss at time t depends on the systemic factor F and the default time cdf H at time t, and **the loss distribution function** expression is:

$$L(t, F) = (1 - R)\Phi\left(\frac{\Phi^{-1}(\text{H(t)}) - \sqrt{\rho}F}{\sqrt{1-\rho}}\right)$$

Collateralized Debt Obligation (CDO)

Pricing of CDO (II/III)

Floating leg market value

The **floating leg market value** of the CDO tranche $[A; D]$ is:

$$\mathrm{JV}^{[A;D]}(0;T) = \int_0^T e^{-rt}\,\mathrm{dEL}_t = e^{-rT}\mathrm{EL}_T + r\int_0^T e^{-rt}\mathrm{EL}_t\mathrm{d}t$$

Fix leg market value

The **fix leg market value** of the CDO tranche $[A; D]$ is:

$$\begin{aligned}\mathrm{JF}^{[A;D]}(0;T) &= s^{[A;D]}\int_0^T e^{-rt}(D - A - \mathrm{EL}_t)\mathrm{d}t \\ &= s^{[A;D]}\mathrm{DV}^{[A;D]}(0;T) \\ &= s^{[A;D]}\left(\left(\frac{D-A}{r}\right)(1-e^{-rT}) - \frac{1}{r}(\mathrm{JV}^{[A;D]}(0;T) - e^{-rT}\mathrm{EL}_T\right)\end{aligned}$$

Collateralized Debt Obligation (CDO)

Pricing of CDO (III/III)

As for CDS, we can use the no arbitrage assumption to calculate the spread of the studied CDO tranche.

Spread of the tranche of a CDO

Thus, the **spread of the CDO tranche** $[A; D]$ is:

$$s^{[A;D]} = \frac{JF^{[A;D]}(0,T)}{DV^{[A;D]}(0,T)}$$

▸ Notebook

Collateralized Synthetic Obligation (CSO)

CSO vs CDO

Cash	**Synthetic**
Large AAA size	Mezzanine AAA+ large super senior
High funding cost	Low funding cost
Limited invested universe	Very large investment universe
Transfer of the assets	Risk transfer only
High management fees	Low management fees
10-15 % high yield	100% investment grade
Average rating BBB-	Average rating A
Low leverage (equity 10 %)	High leverage (2 / 3 %)

Collateralized Synthetic Obligation (CSO)

CSO and CDS indices

Credit Index

iTraxx is a Credit Index used in Europe and Asia with 125 references (the equivalent in the US is **CDX**). It has the following caracteritics:

- Spreads are usually from 10 bp to 120 bp with an average around 35 bp;
- Spread volatility is around 2 bp a day;
- Listed tranches are $[0\%, 3\%]$, $[3\%, 6\%]$, $[6\%, 9\%]$, $[9\%, 12\%]$, $[12\%, 22\%]$;
- Maturities are of 3, 5, 7, 10 years, rolled every 6 months;

▸ iTraxx indice

Implied correlation and base correlation

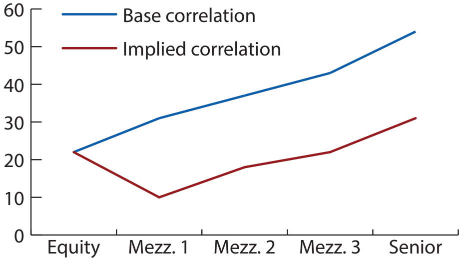
Implied correlation and base correlation

Implied correlation of tranche $[A; D]$

The **implied correlation** of $[A; D]$, knowing the spread of the tranche $s_{A,D}$, is the correlation required in the Vasicek model to price the CSO of tranche $[A; D]$, $s^{[A,D]}$.

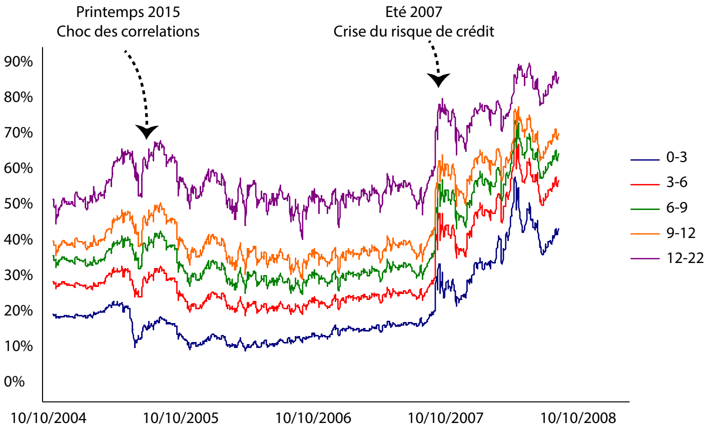
Base correlation

The **base correlation** in K is the implied correlation of $[0; K]$.



Implied correlation and base correlation

Base correlations dynamics



Implied correlation and base correlation

Implied correlation and base correlation – Bijectivity with CDO tranche prices

Bijective relationship between base correlation and CDO tranches spreads

There is a **bijective relationship** between the base correlation and the spread of a CDO tranche :

$$s^{[A;D]} = \frac{JV^{[0;D]}(\rho^{[0;D]}) - JV^{[0;A]}(\rho^{[0;A]})}{DV^{[0;D]}(\rho^{[0;D]}) - DV^{[0;A]}(\rho^{[0;A]})}$$

As option traders usually quote prices with implicit volatilities, CDO traders quote their prices using base correlations.

Implied correlation and base correlation

Implied correlation and base correlation – Interpretation

What do implied and base correlations tell us?

[D'Amato et al., 2005] presents several possible explanations for the correlation smile:

- there is a **segmentation among investors** across tranches;
- the used **models are inefficient**.

Implied correlation and base correlation

Implied correlation and base correlation – Limits of the Vasicek model to price CDO tranches

Wall Street's Math Wizards Forgot a Few Variables

By STEVE LOHR SEPT. 12, 2009

IN the aftermath of the great meltdown of 2008, Wall Street's quants have been cast as the financial engineers of profit-driven innovation run amok. They, after all, invented the exotic securities that proved so troublesome.

▸ The New York Times

Where models the reason for the subprime crisis?

The method used to price CDO tranches has been proven wrong:

- they are too many **homogeneity assumptions** (for correlation, default, maturity, nominal, etc.);
- the **dependence structure** in the model is not extreme enough.

There are a lot of other reasons (quality of the data – Garbage In Garbage Out logic among others) why the subprime crisis happened, most of them will be presented during the Subprimes Crisis Case Study (Lecture 7).

Delta hedging

Delta hedging

A trader wants to buy a protection on a mezzanine tranche;

- He hedges the market value fluctuations of his book by selling protection on individual CDS names;
- Trader's book value is:

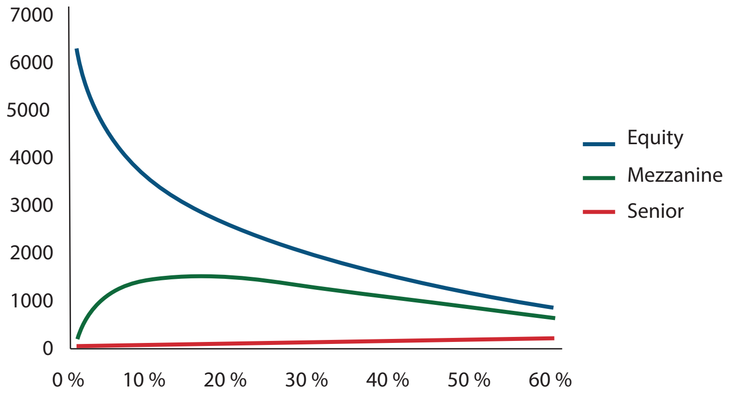
$$P(t) = V_{Tr}(t) + \sum_i \Delta_i V_{CDS_i}(t)$$

Thus, the **hedge ratio** is:

$$\frac{\partial P(t)}{\partial s_j} = 0 \Rightarrow \Delta_j = \frac{\partial V_{Tr}(t)}{DV_j \partial s_j}$$

Hedging single tranche exposure

Pricing sensitivy to correlation



Conclusion
CDO and CSO

- To be filled.

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First-To-Default products – Definition

First-To-Default products – Definition

First-To-Default product

FtD products are similar to CDS contracts except that:

- They are based on a pool of **10 names maximum**;
- The protection buyer pays a **constant spread up to the first default** on the reference basket (if it occurs before maturity);
- When (and if) the first default occurs the protection buyer **delivers the defaulted bond and receives par**.

Would the underlying assets perfectly dependent, the FtD would be equivalent to a single-name CDS.

First-To-Default purpose and arbitrage bounds

First-To-Default purpose and arbitrage bounds

First-To-Default purpose

- They FtD is **riskier than the most risky reference** entity of the basket;
- Buying FtD protection is **cheaper** than buying the protection of each reference name in the basket.

Arbitrage bound of FtD products

Let $(s_1, \dots, s_d)$ be the spreads of the underlying names, we have that the **the spread of the FtD, s_{FtD} arbitrage bounds** are:

$$\max(s_1, \dots, s_d) \leq s_{FtD} \leq \sum_{i=1}^{d} s_i$$

It is a consequence of the no arbitrage asumption.

Rule of thumb for FtD pricing

$$s_{FtD} \approx \frac{2}{3} \sum_{i=1}^{d} s_i$$

Other synthetic products

Other synthetic products (I/II)

Other syntetic products

- **CDO squared**
  - Synthetic CDO on mezzanine synthetic single tranches;
  - More leverage;
  - Caution to systemic risk and overlaps.
- **Leveraged super senior**
  - Super senior tranche leveraged 6-10 times;
  - AAA rating, spread = 60 pb instead of 15 pb;
  - More credit enhancement compared to mezzanine AAA.
- **Combo notes**
  - Combination of A mezzanine and equity;
  - Principal rated A- by the rating agencies.

Other synthetic products

Other synthetic products (II/II)

Other syntetic products

- **EDS: Equity Default Swap**
  - An "equity event" replaces the usual "credit event";
  - The floating leg of the swap pays a cash-flow when the underlying stock hit the threshold of 30% of its value at inception;
  - Need of equity-credit model.
- **CEO: Collateralized Equity Obligation**
  - For example a CDO of EDS or of private equity;
  - In some cases, the maturity of the assets is an issue (ex: private equity).
- **CFO: Collateralized Fund Obligation**
  - CDO collateralized by shares of funds or hedge funds

▸ Quiz

Conclusion
Other credit products

- To be filled.

Conclusion

Portfolio models and Asset Backed Securities

- To be filled.

References I

Benmelech et al. (2009).
The alchemy of CDO credit ratings.
Journal of Monitary Economics.
Link.

Bluhm and Christian (2003).
CDO Modeling: Techniques, Examples and Applications.
HVB.
Link.

Brigo et al. (2010).
Credit Model and the crisis.
Wiley Finance Book.
Link.

Brunel and Roger (2015).
Le Risque de Crédit : des modèles au pilotage de la banque.
Economica.
Link.

References II

Burtschell et al. (2008).
A comparative analysis of CDO pricing models.
laurent.jeanpaul.free.fr.
Link.

D'Amato et al. (2005).
CDS index tranches and the pricing of credit risk correlations.
BIS Quarterly Review.
Link.

Elizalde (2006).
Understanding and Pricing CDOs.
CEMFI.
Link.

Embrechts et al. (1998).
Correlation and Dependency in Risk Management Properties and Pitfalls.
ETH Zurich.
Link.

References III

Frey et al. (2001).
Copulas and credit models.
Univeristy of Zurich.
Link.

Hull and White (2004).
Valuation of a CDO and a nth to default CDS without Monte Carlo Simulation.
Journal Of Derivatives.
Link.

Hull et al. (2009).
The valuation of correlation-dependent credit derivatives using a structural model.
Journal Of Derivatives.
Link.

Joe et al. (1996).
The Estimation Method of Inference Functions for Margins for Multivariate Models.
The University of British Columbia.
Link.

References IV

Laurent and Gregory (2003).
Basket Defaults Swaps, CDO's and Factor Copulas.
laurent.jeanpaul.free.fr.
Link.

Li (2000).
The valuation of Basket Credit Derivatives: A copula function approach.
University of Toronto.
Link.

Plantin (2011).
Good Securitization, Bad Securitization.
Institute For Monetary and Economic Studies - Bank Of Japan.
Link.

Proofs of the lecture

Proofs of the lecture

- Proof 1

Proof – Proof 1