Final Test – 2016-2017

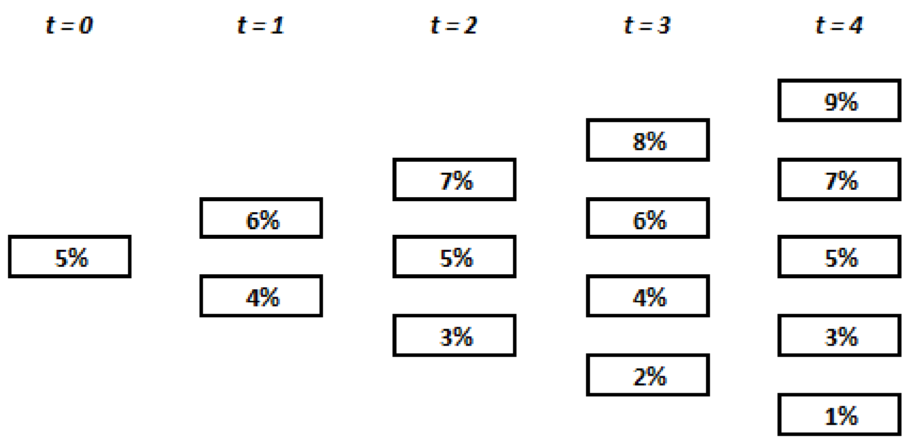
Credit Risk

École Nationale des Ponts et Chausées
Département Ingénieurie Mathématique et Informatique – Master II

Loïc BRIN • Vivien BRUNEL • Benoît ROGER

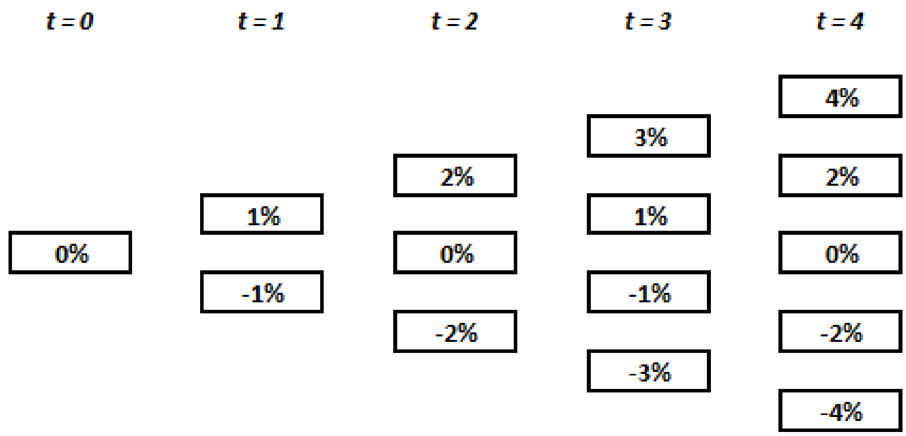
Exercise 1: Computing counterparty risk on an interest rate swap.

We consider a discrete dynamic of interest rates. The date of computation is $t = 0$, and we suppose that the future states of the world are the ones of a binomial tree on four periods, i.e., $t = 0$, $t = 1$, $t = 2$, $t = 3$ and $t = 4$. We suppose that the discount rate is equal to 0 and that the probabilities of reaching the next branches on each knot are both equal to 50%.



1. Fill the tree above with the cash-flows of a swap exchanging a fixed interest rate for a variable one; you pay the fixed rate at 5% and you receive the variable one on a notional of 100.

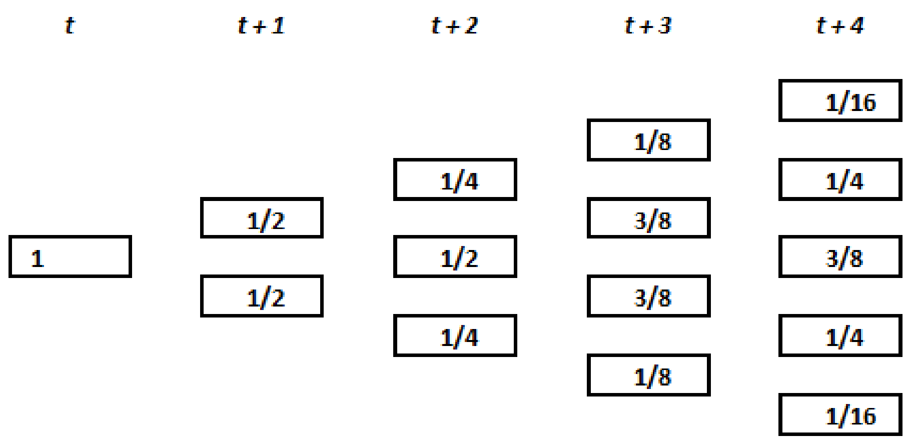
> We express the result in percentages by substracting to the expected cash flow, 5%.The cash-flows are displayed in the tree below:



2. Deduce the market value of the swap at each date.

The market value on each knot is equal to the discounted expected cash flow at maturity ($t = 4$) of the swap.

The probabilities of the two branches coming from a knot being equal and equal to 0.5, the probabilities of each state for each subtree can be easily computed and summarized as follows:



For example, for $t = 0$, there is only one knot and the cash flow is equal to:

- 4% with a likelihood of 6.25%;
- 2% with a likelihood of 25%;
- 0% with a likelihood of 37.5%;
- −2% with a likelihood of 25%;
- −4% with a likelihood of 6.25%.

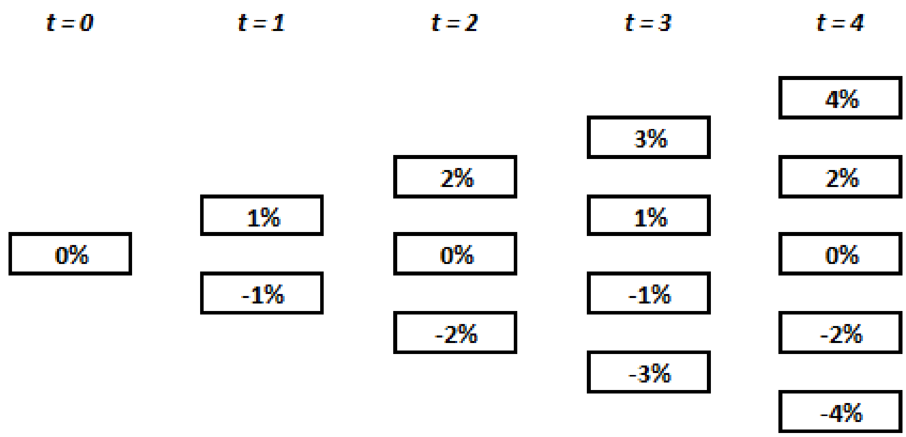
The discount rate being equal to zero, the discount expected cash flow is equal to 0.

In the same fashion the market value at time $t = 2$ on the top knot is give by computing the discount expected cash cash-flows. To do so, we use the subtree starting on the top knot at $t = 2$. The cash flows at maturity ($t = 4$) are as follows:

- 4% with a likelihood of 25%;
- 2% with a likelihood of 50%;
- 0% with a likelihood of 25%;

So the market value on this knot is equale to $4 \times 25\% + 2 \times 50\% + 0 \times 25\% = 2\%$ (expressed in %).

The other market values are displayed in the tree below:



Be careful ! The previous tree appears to be the same as the tree from question 1, but is not computed in the same way.

3. If the counterparty defaults on one of the knot, explain why your maximal credit risk is equal to the positive part of the market value of the swap at this date.

The credit risk born by the owner of the swap towards the swap originator is equal to its replacement cost if its value is positive, and zero, if not. Indeed, would its value be negative, it would mean that the owner owes money to the originator: in that case, the owner does not lose money in any case.

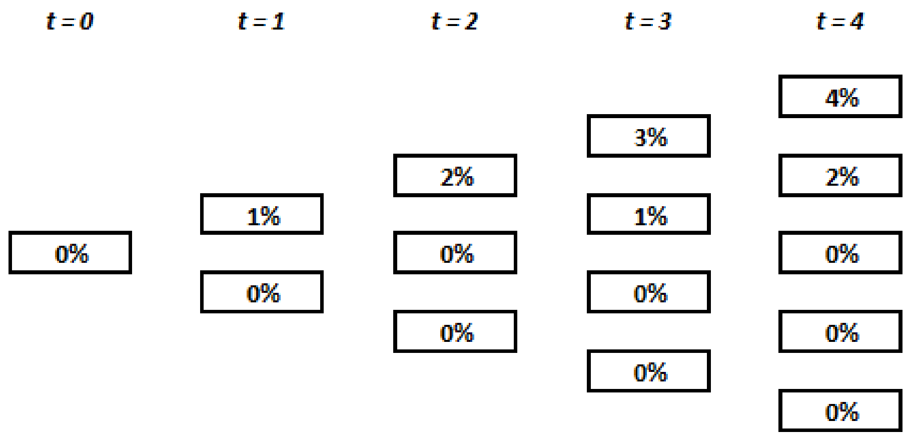
4. We consider we are in $t = 0$. For each date $t > 0$, compute the expectation of the positive part of the market value of the swap. We will call this curve EE(t).

The expected positive part of the market value of the swap, in each t is:

$$EE(t) = \sum_s PP(t,s) \times \mathbb{P}(s)$$

where, $PP(t,s)$ is the positive part in t in the state of the world s, and $\mathbb{P}(s)$ is the probability that the state of the world s happens.

The positive part of the market value is deduced from question 2 and given by the following tree :



Thus, we have;

- $EE(1) = 1\% \times \frac{1}{2} + 0\% \times \frac{1}{2} = 0.50\%$
- $EE(2) = 2\% \times \frac{1}{4} + 0\% \times \frac{1}{2} + 0\% \times \frac{1}{4} = 0.50\%$
- $EE(3) = 3\% \times \frac{1}{8} + 1\% \times \frac{3}{8} + 0\% \times \frac{3}{8} + 0\% \times \frac{1}{8} = 0.75\%$
- $EE(4) = 4\% \times \frac{1}{16} + 2\% \times \frac{1}{4} + 0\% \times \frac{3}{8} + 0\% \times \frac{1}{4} + 0\% \times \frac{1}{16} = 0.75\%$

5. Why can we say that the curve EE(t) corresponds to the future exposures that we have on the counterparty of the swap?

It corresponds to the expected credit risk exposure (positive part of the market value) born by the owner of the swap, towards the originator at each time period.

6. Let us now suppose that the (conditional) default probability of the counterparty at t, knowing that it had not failed at $t-1$, is equal to 10%. Compute the survival probability at $t = 1$, $t = 2$, $t = 3$, and $t = 4$.

The survival probability at $t = 1$ is equal to 90%. At $t = 2$, it is equal to $\mathbb{P}(\tau > 1) \times \mathbb{P}(\tau > 2 \mid \tau > 1) = 0.9 \times 0.9 =$ 81%.

Let $S(t)$ be the survival probability in t, we have:

- $S(0) = 100\%$;
- $S(1) = 90\%$
- $S(2) = 81\%$;
- $S(3) = 72.9\%$;
- $S(4) = 65.61\%$

7. What is the cumulative default probability between $t = 0$ and $t = 4$?

The cumulative default probability between $t = 0$ and $t = 4$ is equal to $1 - 0.6561 = 34.39\%$.

8. We suppose that the recovery rate is equal to 0. What is the expected credit loss on this swap on the whole life of the swap?

Let $P(t)$ denote the probability for the counterparty to default at time t. We have $\forall t > 0,\ \ P(t) = 0.9^{t-1} \times 0.1$ or using the survival probabilities computed in question 6, $\forall t > 0, P(t) = S(t-1) - S(t)$. The expected credit loss on this swap on the whole life of the swap is thus:

$$\begin{aligned} ECL &= \sum_{t=1}^{4} P(t) \times EE(t) \\ &= 10\% \times 0.50\% + 9\% \times 0.50\% + 8.1\% \times 0.75\% + 7.29\% \times 0.75\% \\ &= 0.2104\% \end{aligned}$$

Exercise 2: CDO pricing and tail dependence.

The Vasicek model has been widely used to estimate losses of Asset Backed Securities. We recall that this model is based on the assumption that the assets of the underlying Special Purpose Vehicule are linked together via a common factor through a Gaussian linkage and that the intensity of the latter varies according to the choice of the correlation parameter ρ.

After the subprime crisis (2007-2009), heavy criticisms raised towards this model as it cannot model tail dependencies.

The purpose of this exercise is to understand what tail dependence means, and to study one possible alternative model that takes tail dependencies into account.

Let $\binom{X}{Y}$ be a couple of variables. In the cases that we will study in this exercise, of common *symmetric* distributions for X and Y, the upper tail dependence is equal to:

$$\lambda_u = 2 \times \lim_{x \to \infty} \mathbb{P}(Y > x \mid X = x)$$

1. We consider a couple of variables $\binom{X}{Y} \sim \mathcal{N}(\mathbf{0}; \mathbf{\Sigma})$ with:

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \mathbf{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

a. Compute $\mathbf{P}\binom{X}{Y}$, where: $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ -\rho & 1 \end{pmatrix}$.

$$\mathbf{P}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \\ -\rho X + Y \end{pmatrix}$$

b. Show that:

$$\mathbb{E}(Y \mid X = x) = \rho x \qquad \mathbb{V}(Y \mid X = x) = 1 - \rho^2$$

Let us first notice that $-\rho X + Y$ and X are independent.
We have that:

$$\mathbb{E}(-\rho X + Y \mid X = x) = -\rho x + \mathbb{E}(Y \mid X = x) \tag{1}$$

But we also have that:

$$\begin{aligned} \mathbb{E}(-\rho X + Y \mid X = x) &= \mathbb{E}(-\rho X + Y) \\ &= 0 \end{aligned}$$

as $-\rho X + Y$ and X are independent and the variables are centered.

Thus $\mathbb{E}(Y \mid X = x) = \rho x$.

For the variance, now. We have that:

$$\begin{aligned} \mathbb{V}(-\rho X + Y \mid X = x) &= \mathbb{V}(-\rho x + Y \mid X = x) \\ &= \mathbb{V}(Y \mid X = x) \end{aligned}$$

But we also have that:

$$\mathbb{V}(-\rho X + Y \mid X = x) = \mathbb{V}(-\rho X + Y) \tag{2}$$

as $-\rho X + Y$ and X are independent. And:

$$\begin{aligned} \mathbb{V}(-\rho X + Y) &= \rho^2 \mathbb{V}(X) + \mathbb{V}(Y) - 2\rho \text{Cov}(X, Y) \\ &= \rho^2 + 1 - 2\rho^2 \\ &= 1 - \rho^2 \end{aligned} \tag{3}$$

c. What is the law of $Y \mid X = x$?

$Y \mid X = x$ is Gaussian and we have computed its esperance and variance in question 1.b. Thus $Y \mid X = x \sim \mathcal{N}(\rho x, 1 - \rho^2)$.

2. Deduce from question 1. that $\binom{X}{Y}$ has no upper tail dependence.

We want to compute the upper tail dependence using the formula given at the beginning of the exerice.

$$\begin{aligned} \mathbb{P}(Y > x \mid X = x) =& \bar{\Phi}\left(\frac{x - \rho x}{\sqrt{1-\rho^2}}\right) \\ =& \bar{\Phi}\left(\frac{x\sqrt{1-\rho}}{\sqrt{1+\rho}}\right) \underset{+\infty}{\longrightarrow} 0 \end{aligned}$$

3. We now consider a new couple of variables: $\binom{S}{T} = \sqrt{\frac{\nu}{W}}\binom{X}{Y}$, where $W \sim \chi^2_\nu$ (chi-squared distribution), and $\nu > 0$ is called the degrees of freedom. $\binom{S}{T}$ is said to follow a bivariate Student distribution. It can be shown that $S \mid T = t$ is t-distributed with $\nu + 1$ degrees of freedom and that:

$$\mathbb{E}(S \mid T = t) = \rho t \qquad \mathbb{V}(S \mid T = t) = \left(\frac{\nu + t^2}{\nu + 1}\right)(1 - \rho^2)$$

Compute the upper tail dependence of the Student couple, expressed with $t_\nu(\cdot)$, the cumulative distribution function of the centered reduced t-distribution.

$$\begin{aligned} \mathbb{P}(S > t \mid T = t) =& \bar{t}_\nu\left(\frac{t - \rho t}{\sqrt{1-\rho^2} \times \sqrt{\frac{\nu + t^2}{\nu + 1}}}\right) \\ =& \bar{t}_\nu\left(\frac{\sqrt{1-\rho}\sqrt{\nu+1}}{\sqrt{1+\rho} \times \times \frac{1}{\sqrt{1+\frac{\nu}{t^2}}}}\right) \underset{+\infty}{\longrightarrow} \bar{t}_\nu\left(\frac{\sqrt{1-\rho}\sqrt{\nu+1}}{\sqrt{1+\rho}}\right) > 0 \end{aligned}$$

4. Explain the limits of the Vasicek model in term of dependency modeling and suggest a change to this model that could circumvent this limit.

The Vasicek model is based on a Gaussian linkage. As we have seen in this exercise, Gaussian linkage cannot model tail dependencies. Nonetheless, the occurrence of extreme events on a large part of the assets of a SPV happened during the subprime crisis and could happen again.

Instead of using a Gaussian linkage in the Vasicek model, we could use a Student one as it can model tail dependencies (see question 3.).

Exercise 3: Credit risk, economics and regulation.

1. What are the pros and cons of securitization for banks? for our economies? List two pros, two cons minimum for both banks and our economies.

For banks:

- Pros:
  - less RWA requirements by reducing balance sheet size;
  - much less credit risk to deal with.
- Cons:
  - increase of competition as other firms can do securitization;
  - may increase churn rate as lending to customers reduces attrition.

For our economies:

- Pros:
  - new source of finances as investors can finance new agents;
  - less systemic banks.
- Cons:
  - increase of shadow banking;
  - increase of financial assymetry as the investors are not the ones in relation with the borrowers.

2. In what does regulation push for *originate to distribute* banks? Quickly define the concept and answer the question listing a minimum of three reasons.

We talk about originate to distribute banks when they make loans with the intention of selling them to other institutions and/or investors, as opposed to holding the loans through maturity in their balance sheet as assets.

Such a model is pushed by regulation as it requires increasingly more RWA for each loans contracted by banks and as they are special capital buffers to add for Systemically Important Financial Institutions (SIFI).