Final Test – 2017-2018

Credit Risk

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Exercise 1: Loss Given Default.

1. What is the formula that links the Loss Given Default (LGD) and the Recovery rate (R)? How would you take into account the legal costs (g expressed in percent of the Exposure At Default) due to the default of a counterparty in the previous formula?

> $$\text{LGD} = 1 - R + g$$
>
> where R is the recovery rate and g is the cost resulting from the management of the recovery (expressed as a percentage of the Exposure At Default). The cost resulting from the management of the recovery can be considered as the legal department's yearly budget.

2. Let us consider the loan portfolio of a bank made of 200 000 loans for an average amount of 100 000 EUR. We assume that the probability of default for a one-year horizon is equal to 2% on average and that the recovery rate is 60%. The total yearly budget dedicated to recovery by the legal department is equal to 10 millions EUR. We assume that the 200 000 variables modeling the defaults are i.i.d. The same is assumed for the LDG and the EAD.

a. Compute the Loss Given Default (LGD)

> The total exposure of the portfolio is equal to $N = 200\,000 \times 100\,000 = 20\,000\,000\,000$ EUR. The average loss after recovery is equal to:
>
> $$\begin{aligned} L &= N \times (1-R) \times PD \\ &= 20\,000\,000\,000 \times 0,40 \times 2\% \\ &= 160\,000\,000 \text{ EUR} \end{aligned}$$
>
> and therefore:
>
> $$\begin{aligned} LGD &= \frac{160\,000\,000 + 10\,000\,000}{20\,000\,000\,000 \times 2\%} \\ &= 42.5\% \end{aligned}$$

b. Let us assume that for each loan the Probability of Default (PD), the Exposure At Default (EAD) and the Loss Given Default (LGD) are independent. What is the Expected Loss (EL) of the portfolio at a one-year horizon?

> Since the three parameters for each loans are i.i.d. and assumed independent, we have that:
>
> $$\begin{aligned} EL &= \sum_{i=1}^{200\,000} \mathbb{E}(EAD_i) \times \mathbb{E}(LGD_i) \times PD_i \\ &= 200\,000 \times 100\,000 \times 42.5\% \times 2\% \\ &= 170\,000\,000 \text{ EUR} \end{aligned}$$

c. Let us assume that the default is independent from the LGD and the EAD, but that the correlation between the LGD and the EAD is equal to 0.3. Moreover we will assume that the standard deviation of the LGD of each loan is equal to 0.2 and the standard deviation of the EAD of each loan is equal to 40 000. Compute the Expected Loss (EL) of the portfolio at a one-year horizon.

> In this case, $\mathbb{E}(EAD_i \times LGD_i) \neq \mathbb{E}(EAD_i) \times \mathbb{E}(LGD_i)$. But we know that for any random variables (with a finite second-order moment) X and Y:
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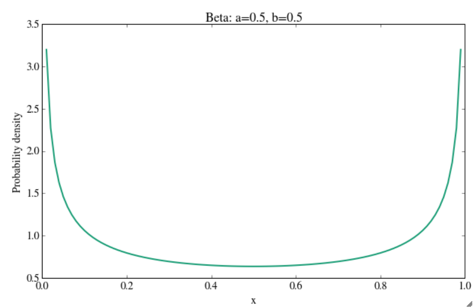
> $$\begin{aligned}\text{Corr}(X,Y) &= \frac{\text{Cov}(X,Y)}{\sigma(X)\times\sigma(Y)} \\ &= \frac{\mathbb{E}(XY)-\mathbb{E}(X)\times\mathbb{E}(Y)}{\sigma(X)\times\sigma(Y)}\end{aligned}$$
>
> and therefore:
>
> $$\mathbb{E}(XY) = \text{Corr}(X,Y)\times\sigma(X)\times\sigma(Y)+\mathbb{E}(X)\times\mathbb{E}(Y)$$
>
> We then get:
>
> $$\begin{aligned}EL &= \sum_{i=1}^{200\,000}\mathbb{E}(EAD_i\times LGD_i)\times PD_i \\ &= \sum_{i=1}^{200\,000}[\text{Corr}(LGD_i, EAD_i)\times\sigma(LGD_i)\times\sigma(EAD_i)+\mathbb{E}(LGD_i)\times\mathbb{E}(EAD_i)]\times PD_i \\ &= 200\,000\times[0.3\times 0.2\times 40\,000+42.5\% * 100\,000]\times 2\% \\ &= 179\,600\,000 \text{ EUR}\end{aligned}$$

3. The LGD is often modeled with a beta distribution which density is u-shaped:



Explain why.

> For most of the loans at default the bank usually recover either almost the total amount of the exposure, or almost nothing.

Exercise 2: A Merton-like approach to the Vasicek model.

In class, we explore two ways to define the Vasicek model: the first one is based on latent returns R_i and the second one on an intensity-based assumption for individual default with a Gaussian copula linkage.

The purpose of this exercise is to show that the Vasicek model can be seen as an extension of the Merton model.

1. Let us first consider the default on only one counterparty. As in the Merton model, we will assume that the default occurs when the value of equity is smaller that a threshold. We assume that the diffusion of the value of equity verifies:

$$dV_t = \mu V_t dt + \sigma V_t d\tilde{B}_t + \beta V_t dB_t$$

where:

- V_t denotes the company's equity value at time t;

- σ is a constant that denotes the sensitivity of the firm to the systemic risk;
- μ is a constant that denotes the return rate of the firm's equity;
- β denotes the sensitivity of the firm to the idiosyncratic risk;
- B_t denotes the standard Wiener process associated to the idiosyncratic risk;
- $\tilde{B}_t$ denotes the standard Wiener process associated to the systemic risk.

B_t and $\tilde{B}_t$ are independent.

Using Ito's lemma, one can show that:

$$V_1 = V_0 \exp\left(\mu - \frac{1}{2}(\sigma^2 + \beta^2) + \beta B_1 + \sigma \tilde{B}_1\right)$$

where $B_1 \sim \mathcal{N}(0,1)$, $\tilde{B}_1 \sim \mathcal{N}(0,1)$ and B_1 and $\tilde{B}_1$ are independent.

a. We denote by D the variable equal to one if a default occurs. We assume that default occurs if $V_1 < m$. Show that

$$PD = \Phi\left(\frac{\log \frac{m}{V_0} - \mu + \frac{1}{2}(\sigma^2 + \beta^2)}{\sqrt{\sigma^2 + \beta^2}}\right)$$

> D is a Bernoulli distributed random variable, $D \sim \mathcal{B}(PD)$. The default occurs, $D = 1$, if the level V_1 stands below the level of liabilities m. Thus, we can write:
>
> $$\begin{aligned} PD =& \mathbb{Q}\left(V_0 \exp\left(\mu - \frac{1}{2}(\sigma^2 + \beta_V^2) + \beta B_1 + \sigma \tilde{B}_1\right) < m\right) \\ PD =& \mathbb{Q}\left(\mu - \frac{1}{2}(\sigma^2 + \beta_V^2) + \beta B_1 + \sigma \tilde{B}_1 < \log \frac{m}{V_0}\right) \\ PD =& \mathbb{Q}\left(\beta B_1 + \sigma \tilde{B}_1 < \log \frac{m}{V_0} - \mu + \frac{1}{2}(\sigma^2 + \beta_V^2) +\right) \end{aligned}$$
>
> We know that B_1 and $\tilde{B}_1$ are mutually independent and $\mathcal{N}(0,1)$ distributed, so we know that $\beta B_1 + \sigma \tilde{B}_1$ is $\mathcal{N}(0, \sigma^2 + \beta^2)$ distributed.
>
> We can thus standardize the random variable in the right hand side of the equality and get:
>
> $$\begin{aligned} PD =& \mathbb{Q}\left(\frac{\beta B_1 + \sigma \tilde{B}_1}{\sqrt{\sigma^2 + \beta^2}} < \frac{\log \frac{m}{V_0} - \mu + \frac{1}{2}(\sigma^2 + \beta_V^2)}{\sqrt{\sigma^2 + \beta^2}}\right) \\ PD =& \Phi\left(\frac{\log \frac{m}{V_0} - \mu + \frac{1}{2}(\sigma^2 + \beta^2)}{\sqrt{\sigma^2 + \beta^2}}\right) \end{aligned}$$

b. We denote $\rho = \frac{\sigma^2}{\sigma^2 + \beta^2}$, show that:

$$\mathbb{Q}(D = 1 \mid \tilde{B}_1 = y) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}\, y}{\sqrt{1-\rho}}\right)$$

> We can write:
>
> $$\begin{aligned} \frac{\beta B_1 + \sigma \tilde{B}_1}{\sqrt{\sigma^2 + \beta^2}} =& \frac{\sigma \tilde{B}_1}{\sqrt{\sigma^2 + \beta^2}} + \frac{\beta B_1}{\sqrt{\sigma^2 + \beta^2}} \\ =& \sqrt{\rho}\tilde{B}_1 + \sqrt{1-\rho} B_1 \end{aligned}$$

We thus have:

$$PD = \mathbb{Q}\left(\frac{\beta B_1 + \sigma \tilde{B}_1}{\sqrt{\sigma^2+\beta^2}} < \frac{\log\frac{m}{V_0} - \mu + \frac{1}{2}(\sigma^2+\beta_V^2)}{\sqrt{\sigma^2+\beta^2}} + \right)$$

which is equivalent to:

$$\Phi^{-1}(PD) = \left(\frac{\log\frac{m}{V_0} - \mu + \frac{1}{2}(\sigma^2+\beta_V^2)}{\sqrt{\sigma^2+\beta^2}}\right)$$

That means that, $D = 1$ if $\sqrt{\rho}\tilde{B}_1 + \sqrt{1-\rho}B_1 \leq \Phi^{-1}(PD)$. Thus:

$$\begin{aligned} Q(D=1 \mid \tilde{B}_1 = y) =& \mathbb{Q}(\sqrt{\rho}\tilde{B}_1 + \sqrt{1-\rho}B_1 \leq \Phi^{-1}(PD) \mid \tilde{B}_1 = y) \\ =& \mathbb{Q}(\sqrt{\rho}y + \sqrt{1-\rho}B_1 \leq \Phi^{-1}(PD)) \\ =& \mathbb{Q}\left(B_1 \leq \frac{\Phi^{-1}(PD) - \sqrt{\rho}\tilde{y}}{\sqrt{1-\rho}}\right) \end{aligned}$$

As we know that, $B_1 \sim \mathcal{N}(0,1)$, we can conclude that:

$$\mathbb{Q}(D=1 \mid \tilde{B}_1 = y) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)$$

c. What interpretation can you make of ρ?

ρ is the *proportion* of the risk that is carried by a systemic factor. This risk is the same for all the considered firms : it is a correlation parameter.

2. We want to model the loss distribution of a portfolio of bonds, and denote L the variable equal to the percentage of the portfolio that is lost after one year. We assume that the recovery rate is 0 and that the default of each name in the portfolio occurs as modeled in 1. Eventually, we assume the portfolio is infinite homogeneous.

a. Can you remind what infinite homogeneous means in that case?

- Infinite: it means that we assume there is a infinity (countable) of bonds on equal infinitesimal size (same nominal) in our portfolio;
- Homogeneous: it means that we assume they have (i) the same maturity (one year in our case), (ii) the same probability of default, and (iii) the same recovery rate (equal to zero in our case).

b. Show that

$$\forall \theta \in [0;1], \qquad \mathbb{Q}(L < \theta) = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(\theta) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right)$$

$$\begin{aligned} L \mid \tilde{B}_1 &= \lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{V_1 < m\}} \\ &= \lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\left\{B_1 < \frac{\Phi^{-1}(PD) - \sqrt{\rho}\tilde{B}_1}{\sqrt{1-\rho}}\right\}} \\ &\underbrace{=}_{\substack{\text{Law of} \\ \text{large numbers}}} \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}\tilde{B}_1}{\sqrt{1-\rho}}\right) \end{aligned}$$

For a given frequency of losses, θ, we can calculate the corresponding value of the market shock, $\tilde{B}_1(\theta)$ that will produce exactly the level of loss.

$$\theta = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}\tilde{B}_1(\theta)}{\sqrt{1-\rho}}\right)$$

So we have:

$$\Phi^{-1}(\theta) = \frac{\Phi^{-1}(PD) - \sqrt{\rho}\tilde{B}_1(\theta)}{\sqrt{1-\rho}}$$

And then:

$$\tilde{B}_1(\theta) = \frac{\Phi^{-1}(PD) - \sqrt{1-\rho}\Phi^{-1}(\theta)}{\sqrt{\rho}}$$

Since the proportion of portfolio losses decreases with $\tilde{B}_1$, the probability that the proportion of loans that defaults (L) is less that θ is:

$$\mathbb{Q}(L < \theta) = \mathbb{Q}(\tilde{B}_1 > \tilde{B}_1(\theta)) = \mathbb{Q}\left(\tilde{B}_1 > \frac{\Phi^{-1}(PD) - \sqrt{1-\rho}\Phi^{-1}(\theta)}{\sqrt{\rho}}\right)$$

And thus:

$$\mathbb{Q}(L < \theta) = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(\theta) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right)$$

3. Conclude by comparing this model with the Vasicek model.

The loss distribution of this model is the same as in the Vasicek model. We conclude that the Vasicek model can be seen as a portfolio extension of the Merton model with two Wiener processes: one shared for all the the firms diffusions and an independent one different for each firm.

Exercise 3: The subprime mortgage crisis: a model risk crisis?

1. "*The formula that killed Wall Street*", "*Mark-to-myth*", "*Mea Copula*", what are these journalistic quotes referring to?

We saw in class that the Vasicek model is equivalent to an intensity based model with Gaussian copula linkage. As Gaussian copulas cannot model tail dependence, they were not able to anticipate / price extreme phenomena as the ones that happen from 2007 to 2009 (high number of defaults on mortgages granted to subprime borrowers). For this reason, they have been identified as, among other things, responsible for the crisis.

2."*The problem is not that mathematics was used by the banking industry, the problem was that it was abused by the banking industry. Quants were instructed to build models which fitted the market prices. Now if the market prices were way out of line, the calibrated models would just faithfully reproduce those wacky values, and the bad prices get reinforced by an overlay of scientific respectability!*" wrote L.C. Rogers from the University of Cambridge.
Can you be more specific about what L.C. Rogers is referring to?

The implicit and based correlations extracted from CSO prices are supposed to be equal for all the tranches if ones assumes that the Vasicek model is applicable: in practice, it is not the case. As for the Black & Scholes model, one needs to adapt the parametrization of the model so that it fits market prices in order to avoid an arbitrage situation.

3. According to you, what best practices can mitigate model risk?

- Independent validation team;
- Use of alternative models;
- Use of stress testing;
- Test the robustness of the model, the sensititvity of the parametrs;
- Control the quality of the data that is used (data lineage control for example), etc.

4. What other reasons have been identified as responsible for the subprime mortgage crisis?

Among others, the Inquisit Reports cites:

- Product opaqueness and poor quality of the transmitted information and used data;
- Regulatory failure – no skin in the game;
- Excessive ownership planning and incentives by the government;
- Absence of real challenges by rating agencies.