Final Test – 2019-2020

Credit Risk

École Nationale des Ponts et Chaussées
Département Ingénieurie Mathématique et Informatique – Master II

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Exercise 1: Large pool model and bank steering (8 points).

In this exercise, we are going to review notions on portfolio models and the benefit of risk based steering in a bank. The first part of the exercise is dedicated to the *large pool model* which is close to the portfolio model we have studied in class. The second part is about bank steering in a large pool model framework.
As a specialized consumer finance entity, we may invest in two different portfolios of assets (e.g. car loans and mortgages).
Each portfolio ($P^{(i)}$) (where i=1,2) is made of an infinite number of loans, with characteristics ($PD^{(i)}$), ($LGD^{(i)}$). The systemic risk factors are ($F^{(i)}$) and the internal correlations between assets returns are ($\rho^{(i)}$). Correlation between both systemic factors is ρ. Finally, the spread of each portfolio are ($s^{(i)}$).

Part 1 (Course).

1. Compute the expected loss and the standard deviation of losses on a pool.

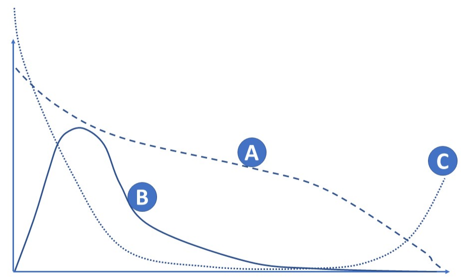
> $EL = PD \times LGD$ and StdDev= $LGD \times \sqrt{\Phi_2(s,s,\rho) - PD(1-PD)}$, where $s = \Phi^{-1}(PD)$ and Φ is the cumulative distribution function of a normal standard variable, Φ_2 the covariate distribution function.

2. Derive the loss distribution function (assumption to simplify the computation: LGD(i)=100%).

> We recall that $Pr[L \leq l] = Pr[F \geq f_l]$ where f_l such that $l = LGD\Phi\left(\frac{s-\sqrt{\rho}f_l}{\sqrt{1-\rho}}\right)$, i.e. $f_l = \frac{s-\sqrt{1-\rho}\Phi^{-1}(l)}{\sqrt{\rho}}$.
> This leads to $Pr[L \leq l] = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(l)-s}{\sqrt{\rho}}\right)$.
> Deriving w.r.t. l, we obtain:
>
> $$\frac{dPr[L \leq l]}{dl} = \sqrt{\frac{1-\rho}{\rho}}\left(\Phi^{-1}\right)'(l)\Phi'\left(\frac{\sqrt{1-\rho}\Phi^{-1}(l)-s}{\sqrt{\rho}}\right),$$
>
> which finally leads to:
>
> $$\frac{dPr[L \leq l]}{dl} = \sqrt{\frac{1-\rho}{\rho}}\exp\left(-\frac{1}{2\rho}\right)\exp\left(-\frac{1}{2\rho}\left[(1-2\rho)\Phi^{-1}(l)^2 + 2\sqrt{1-\rho}\Phi^{-1}(l)\right]\right)$$

3. In the following graph, three different shapes of loss distribution functions are presented. Please explain which curves correspond to:

- A correlation between 0 and 50%;
- A correlation equal to 50%;
- A correlation above 50%.



Curve B corresponds to a correlation below 50%, curve A occurs with a correlation equal to 50%. Curve C corresponds to a correlation above 50%.

4. Suppose for simplification that the new loss distribution, after investing x_i in $P^{(i)}$ follows a Vasicek law. What are the parameters of the Vasicek law? (match expected loss and standard deviation)

The parameters that we need to calibrate are: PD, LGD and rho.

The average PD is equal to $\dfrac{x_1 PD^{(1)} + x2PD^{(2)}}{x_1 + x_2}$.
The Expected Loss is also easy to compute: $EL = PD \times LGD \times (x_1 + x_2) = x_1\ PD^{(1)}\ LGD^{(1)} + x_2\ PD^{(2)}\ LGD^{(2)}$, and from it, we can deduce LGD.
Concerning ρ, we know that $\sigma^2(L) = x_1^2\sigma^2(L^{(1)}) + x_2^2\sigma^2(L^{(2)}) + 2x_1x_2\,\mathrm{Corr}(L^{(1)}, L^{(2)})$.

Additionally, $\mathrm{Corr}(L^{(1)}, L^{(2)}) = (x_1 PD^{(1)} LGD^{(1)}) \times (x_2\ PD^{(2)}\ LGD^{(2)}) - x_1x_2\mathbb{E}(L_1L_2)$.
The only term we do not know is $\mathbb{E}(L_1L_2)$, which we compute the following way:

$$\mathbb{E}\left[\left(\frac{1}{N_1}\sum_{n_1=1}^{N_1}\mathbb{1}_{\{R_{n_1}^{(1)}<s_1\}}\right)\left(\frac{1}{N_2}\sum_{n_2=1}^{N_2}\mathbb{1}_{\{R_{n_2}^{(2)}<s_2\}}\right)\right] = \Phi_2(s_1, s_2, \sqrt{\rho_1\rho_2})$$

Letting $N_1, N_2 \to +\infty$, we obtain :

$$\mathrm{Corr}(L^{(1)}, L^{(2)}) = (x_1\ PD^{(1)}\ LGD^{(1)}) \times (x_2\ PD^{(2)}\ LGD^{(2)}) - x_1x_2LGD^{(1)}LGD^{(2)}\Phi_2(s_1, s_2, \sqrt{\rho_1\rho_2}).$$

5. The level of capital that is targeted by the bank is the 99.9% Expected Shortfall (that is the Expected Loss, knowing that the loss is above the 99.9% quantile).

- Express the 99.9% quantile for each portfolio;
- Derive the 99.9% Expected Shortfall.

We have seen in class that $\mathrm{Quantile}(99.9\%) = LGD\Phi\left(\dfrac{\Phi^{-1}(PD) + \sqrt{\rho}\Phi^{-1}(99.9\%)}{\sqrt{1-\rho}}\right)$, and we know that the expected shortfall is computed as the limit (for $N \to \infty$) of $\dfrac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[\mathbb{1}_{\{R_n<s\}}\mathbb{1}_{\{F<-\Phi(99.9\%)\}}\right]$, which leads to:

$$\mathbb{ES}(99.9\%) = \frac{1}{0.01\%}\left[\Phi_2(\Phi^{-1}(PD), -\Phi^{-1}(99.9\%), \sqrt{\rho}) - 0.01\% \times \mathrm{Quantile}(99.9\%)\right]$$

Part 2 (Bank Steering).

6. The level of capital targeted by the bank is called the Economic Capital and depends on the level x_1 invested in $P^{(1)}$ and x_2 in $P^{(2)}$.
The CEO announces to the market a target Return On Equity of 10%. Knowing that we may borrow debt at a null price, what is the average cost of capital?

> The Cost of Equity is then of 10%, and the Cost of Debt is of 0%. Thus, the WACC is equal to $\frac{EC \times 10\%}{EC + Debt}$.

7. The amount of equity the bank has is EC (its Economic Capital). Show that the EVA (Economic Value Added), as a function of x_1 and x_2 is:

$$EVA = x_1 s^{(1)} - x_1 PD^{(1)} LGD^{(1)} + x_2 s^{(2)} - x_2 PD^{(2)} LGD^{(2)} - 10\% \times EC \tag{1}$$

> $EVA = (Spread - EL) - 10\% \times EC$. $Spread - EL$ can be easily computed on each portfolio: for the first portfolio, it is equal to $x_1 s^{(1)} - x_1 PD^{(1)} LGD^{(1)}$. Thus, the formula comes naturally.

We are now going to find the optimal mix under simplification assumptions.

8. Expected shortfall is a coherent measure. It implies that for each $\mu > 0$, $ES(\mu x_1, \mu x_2) = \mu ES(x_1, x_2)$.
Show that the ES satisfies the Euler condition:

$$ES(x_1, x_2, \ldots x_N) = \sum_i x_i \frac{\partial ES}{\partial x_i} \tag{2}$$

$x_i \frac{\partial ES}{\partial x_i}$ is then the contribution of activity i to global Expected Shortfall (ES).

> As the Expected Shortfall is a coherent measure, for each real number μ, $ES(\mu x_1, \mu x_2) - \mu ES(x_1, x_2) = 0$. Taking the derivative w.r.t. μ, we find formula (2).

9. We now have a given amount of equity to invest EC_{Max}. Show that the optimization problem is:

$$\max_{(x_1, x_2)} EVA = x_1 s^{(1)} - x_1 PD^{(1)} LGD^{(1)} + x_2 s^{(2)} - x_2 PD^{(2)} LGD^{(2)} - 10\% \times ES \; ; s.t. ES \leq EC_{Max} \tag{3}$$

In order to solve the problem, we define the Lagrangian as:

$$L(x_1, x_2, \lambda) = EVA + \lambda(EC_{Max} - ES(x_1, x_2)).$$

The Kuhn-Tucker conditions (that are satisfied) imply that:

$$\frac{\partial L}{\partial x_i} = 0, \; i = 1, 2$$

$$\lambda (EC_{Max} - ES(x_1, x_2)) = 0 \text{ and } \lambda \geq 0.$$

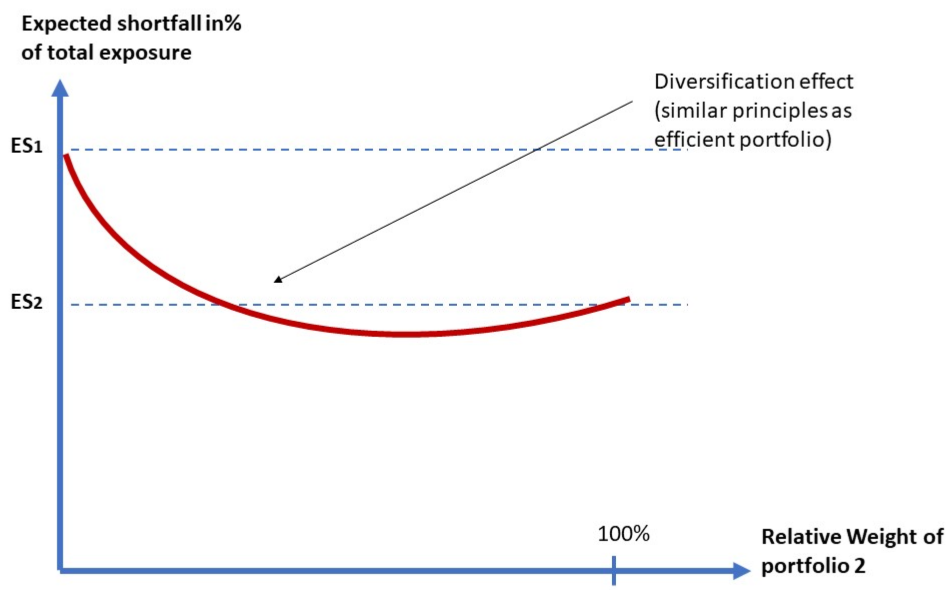
> We need to solve the following optimization problem: maximize EVA, under the constraint that computed expected shortfall remains under available capital, which leads to the reply.

10. Show that at the optimal point, RAROC on portfolio 1 and portfolio 2 are equal (and at least equal to 10%).

> Clearly because of the previous property, the constraint on capital is binding. Let us yet solve the optimization problem, using formula (1), proved in question 7: $\frac{\partial L}{\partial x_i} = 0$ implies that $\left(s^{(i)} - PD^{(i)} LGD^{(i)}\right) - (10\% + \lambda) \frac{\partial ES}{\partial x_i} = 0$.
> Equivalently, for $x_i > 0$:
>
> $$\frac{x_i s^{(i)} - x_i PD^{(i)} LGD^{(i)}}{x_i \frac{\partial ES}{\partial x_i}} = 10\% + \lambda.$$
>
> Thanks to question 8., we find that the first term of the equation is the RAROC on portfolio i, and thus that all portfolio have the same RAROC (equal to $10\% + \lambda$).

11. (Bonus - 2 points) Another important property of a coherent risk measure is sub-additivity, i.e. $ES(P1+P2) < ES(P1)+ES(P2)$. Explain why through diversification, we reached at optimum a better return than on each of the initial portfolios (we may see that the expected shortfall is a convex function on (x_1, x_2)).
Another way to say it: through adequate risk management, we could either increase our profitability, either reduce the margin (the spread) so as to be even more competitive while maintaining an adequate level of profitability (if we did not increase our own costs).

The sub-additivity implies that the ES measure is convex, as illustrated below:



Exercise 2: Transition matrix (6 points).

$$A = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Let A denote a simplified transition matrix.
1.Which row of the matrix A corresponds to the default state? Justify your answer.

We can see that the third row is an absorbing state (a borrower defaulting cannot transition to another state/rating), which corresponds to the *default* state in a transition matrix.

2. Show that 1 is an eigenvalue of any stochastic matrix (each row sums to 1) and deduce that any stochastic matrix to the power n ($n \in \mathbb{N}$) is also stochastic. Determine the eigenvector of the eigenvalue 1.

We know that each row of a stochastic matrix sums to 1. Let M denote a stochastic matrix, this property is equivalent to:

$$M \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, $(1,1,1)^t$ is an eigenvector of M and we have that:

$$\forall n \in \mathbb{N}, M^n \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (1)^n \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

M^n is therefore a stochastic matrix (each row of M^n sums to 1).

3. Show that $(1,1,0)^t$ and $(-2,1,0)^t$ are eigenvectors of A and determine the corresponding eigenvalue(s).

$$A \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix} = \frac{4}{5} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
$$A \times \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-2}{5} \\ \frac{1}{5} \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$\frac{4}{5}$ and $\frac{1}{5}$ are therefore eigenvalues of A with the respective eigenvectors $(1,1,0)^t$ and $(-2,1,0)^t$.

4. Prove that A can be written as follows:

$$A = P \times \Omega \times P^{-1}$$

where Ω is a 3×3 diagonal matrix.

A is 3×3 matrix with three eigenvalues 2, $\frac{4}{5}$ and $\frac{1}{5}$. A is therefore diagonalizable and can be written as follows:

$$A = P \times \Omega \times P^{-1}$$

where,

$$\Omega = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

with P the matrix composed of the eigenvectors of A:

$$P = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

P^{-1} is obtained by inverting P and we have:

$$P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{3} & \frac{2}{3} & -1 \\ \frac{-1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

5. Determine the probability that a borrower with a rating corresponding to the first line of A will default within the next 20 years.

$$\forall n \in \mathbb{N}, \quad A^n = \left(P \times \Omega \times P^{-1}\right)^n = P\Omega^n P^{-1}$$

Since Ω is diagonal:

$$\Omega^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{4}{5}\right)^n & 0 \\ 0 & 0 & \left(\frac{1}{5}\right)^n \end{bmatrix}$$

The probability that a borrower with a rating corresponding to the first line of A has a probability of defaulting within the next 20 years equal to the third element of the matrix A^{20} : $1-\left(\frac{4}{5}\right)^n$.

Exercise 3: Multiple choice questions (6 points).

There might be several correct answers. You will get 0.5 points each time you answer correctly to one of the 12 questions. You will not lose points if your are wrong.

1. Total SA has a bond quoted on the market with a spread of 42 bps. It also has a CDS whose annual premium is of 38 bps. Total SA Fitch rating is A, which corresponds to a probability of default of 0.18 %. We assume here that the recovery rate is equal to 0%. Which of the following statements are correct?

1. The market is not the true reflection of Total SA's credit risk;
2. The real world probability of default of Total SA is 18 bps;
3. The risk neutral probability of default of Total SA is 42 bps;
4. The risk neutral probability of default of Total SA is 38 bps.

2 and 4.

2. Which of the following statements are correct?

1. Credit risk is mitigated and managed through diversification in retail banking;
2. Credit risk is mitigated and managed partially through hedging in market activities and the CVA desk takes charge of it;
3. Securitization is a way to mitigate credit risk in retail banking;
4. CDS can be used to mitigate credit risk in retail banking.

1, 2 and 3.

3. A given bank's liabilities are the following: 10% is equity, 40% is long terme debt (interest rates at 3% annually), and the rest are deposits on which they are no interests paid to the customers. Shareholders expect 10% of ROE. Which of the following statements are correct?

1. The WACC is 10%;
2. The WACC is 4.4%;
3. The WACC is 2.2%;
4. A project with Return on Investment at 2.7% should be launched as it is profitable.

3 and 4.

4. In Merton's model, one can say:

1. That shareholders have a call on the company performance;
2. That debtors are long a put on the company performance;
3. That debtors are short a put on the company performance;
4. That shareholders would rather have a CEO which takes risks.

1, 3 and 4.

5. Among others, the 2007-2009 subprimes crisis can be explained by:

1. Information asymmetry;
2. Data quality;
3. Arbitrage opportunities;
4. Models' quality.

1, 2 and 4.

6. It happens that there is a spread between a CDS quote and a bond quote. Why?

1. Because the definition of the default is often not the same in both contracts;
2. Because of counterparty risk;
3. Because of risk aversion;
4. Because of the use of risk neutral probability to price the CDS.

1 and 2.

7. When would a deal generate no CVA?

1. When its market value is null at inception;
2. When it is perfectly collateralized;
3. When both counterparties are States;
4. When a netting agreement has been signed.

2.

8. Morgan Brothers has stroke a USD/EUR forex swap deal with the World Bank (the World Bank will receive USD). The WB won't collateralize but MB will, and will adjust its post every month. Which of the following statements are correct?

1. The WB generates counterparty risk for MB at inception;
2. MB generates counterparty risk for WB at inception;
3. None of the previous statements is correct.

3.

9. What about CVA at inception in this same deal (the WB and MB deal):

1. The WB generates CVA for MB at inception;
2. MB generates CVA for the WB at inception;
3. None of the previous statements is correct.

1 and 2.

10. The following week, the USD becomes stronger. Which of the following statements are correct?

1. The WB generates additionnal counterparty risk for MB;
2. MB generates additional counterparty risk for the WB;
3. None of the previous statements is correct.

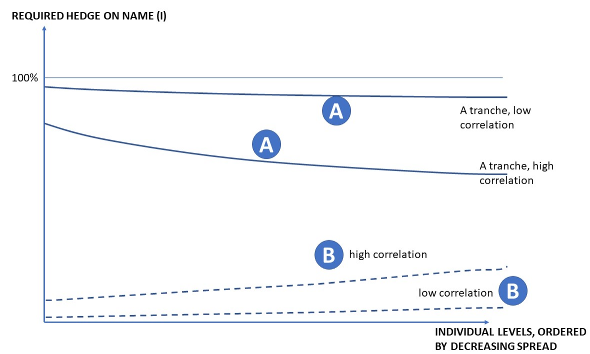
| 2.

11. What about CVA the following week in this same deal:

1. CVA on the WB for MB increases;
2. CVA on the WB for MB decreases;
3. CVA on MB for the WB decreases;
4. CVA on MB for the WB increases.

| 2 and 4.

12. You have the possibility to invest in CDO tranches. The following graph represents the hedging quantity you need to buy to protect yourself against individual spreads movements.
Which curve corresponds to the equity tranche, which curve corresponds to a senior tranche?



1. The equity tranche is A;
2. The equity tranche is B.

| 1.