Credit Risk
Lecture 3 – Structural Models

Loïc BRIN

École Nationale des Ponts et Chaussées
Département Ingénierie Mathématique et Informatique (IMI) – Master II
Objectives of the lecture

Teaching objectives

At the end of this lecture, you will:

▶ Understand the **principles of structural approaches** in credit risk;
▶ Know how to **compute the equity and debt values of a firm** under the Merton model’s assumptions;
▶ Be able to compute the **Merton probability of default** of a firm;
▶ Know how to derive the **optimal amount of debt** for a firm’s investors from the Leland model;
▶ Be aware of the **limitations of structural approaches**.
1. The structural models – Prerequisites

2. Merton Model, 1973

3. Leland Model, 1994
1 - Prerequisites
Accounting basics
Balance sheet structure

Firms finance their Assets with Liabilities:
- Equity;
- Debts.

All of this is summarized in their balance sheet.

Structural model are based on the structure of the liabilities of the firm.
Option theory basics

Call option payoff

Source: www.brilliant.org/wiki/call-and-put-options/
Option theory basics

Put option payoff

Source: www.brilliant.org/wiki/call-and-put-options/
2 - Merton Model, 1973
**Merton Model – Framework**

What are the main assumptions?

**Balance sheet is equilibrated.** If the value of assets changes, so do the liabilities. Let us suppose that the debt is a zero-coupon bond of maturity $T$.

↓

If assets value is inferior to the debt nominal, that is, if equity is inferior to 0: the firm is in default.

↓

In case of liquidation, bonds and loans investors expect to recover the nominal of the debt ($D$), and **equity holders get what remains**.

<table>
<thead>
<tr>
<th>Value of Assets ($V_T$)</th>
<th>Shareholder’s flow</th>
<th>Debt holders’s flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_T \geq D$</td>
<td>$V_T - D$</td>
<td>$D$</td>
</tr>
<tr>
<td>$V_T &lt; D$</td>
<td>0</td>
<td>$V_T$</td>
</tr>
</tbody>
</table>
Merton Model – Framework

The value of the debt

Under the risk-neutral probability, the debt value at t is equal to the expected discounted cash flows from the debt at maturity $T$:

$$D_t = \mathbb{E}^Q \left[ e^{-r(T-t)} \min (D, V_T) \mid \mathcal{F}_t \right]$$

$$= \mathbb{E}^Q \left[ e^{-r(T-t)} D \mid \mathcal{F}_t \right] - \mathbb{E}^Q \left[ e^{-r(T-t)} (D - V_T)^+ \mid \mathcal{F}_t \right]$$

Hence, the value of the debt is equal to the price of a risk-free zero-coupon of maturity $T$ minus the value of a put on the value of the assets of maturity $T$ and strike $D$. 
Merton Model – Framework

The value of the equity

Under the risk-neutral probability, the **equity** value at t is equal to the **expected discounted cash flows of the shareholders**:

\[
E_t = \mathbb{E}^Q \left[ e^{-r(T-t)} \max (V_T - D, 0) \mid \mathcal{F}_t \right]
\]

The value of the **equity** is then equal to the price of **call on the firm assets** of maturity \(T\) and strike \(D\).
Merton model – Main results

Diffusion of asset’s value

Diffusion of Asset’s value in Merton model

Let \((V_t)_t\) be the process modeling the value of the firm. In Merton model, we have:

\[
\frac{dV_t}{V_t} = rt + \sigma d\tilde{W}_t
\]

where \(\tilde{W}_t\) denote a standard brownian motion under the risk-neutral probability and \(r\) the risk-free rate.

The asset’s value of the firm is modelled with a geometric brownian motion.
Merton model – Main results

How to get the value of the debt and equity?

**Value of debt and equity – Black-Scholes results**

Let $D$ be the amount of debt in the balance sheet in $t$, $D_t$ its value, and $E_t$ the value of equity (in $t$). We have:

$$D_t = D e^{-r(T-t)} N(d_2) + V_t N(-d_1)$$

$$E_t = V_t N(d_1) - D e^{-r(T-t)} N(d_2)$$

with:

$$d_1 = \frac{\log \frac{V_t}{D} + \left( r + \frac{\sigma^2}{2} \right)(T-t)}{\sigma \sqrt{T-t}}$$

$$d_2 = d_1 - \sigma \sqrt{T-t}$$

Since the value of the firm follows a geometric brownian motion, we can derive the **debt and equity values** using the **Black-Scholes** formula.
Merton model – Main results
Can we derive the Debt's spread?

The debt spread \( s_t \) of the debt \( D \) in \( t \) is the **actual interest rate minus the risk-free rate** \( r \), so that \( D_t = De^{-(r+s_t)(T-t)} \).

**Spread – Value**

In Merton model, the spread is then equal to:

\[
    s_t = \frac{1}{T-t} \log \left( \frac{D}{D_t} \right) - r
\]

where \( D_t = De^{-r(T-t)}N(d_2) + V_tN(-d_1) \)
Merton model – Main results

What does the Call-Put parity mean in Merton model?

**Call-Put parity**

The Call-Put parity corresponds to an obvious *equation from corporate finance.*

\[
\text{Assets} = \text{Equity} + \text{Debt}
\]

Reminder: Let \( A_t \) denote the value of an asset in \( t \) and \( C_t \) and \( P_t \) denote the value of a European call option and a European put on the underlying \( A_t \) with strike \( D \). The *classical call-put parity* is given by:

\[
C_t - P_t = A_t - De^{-r(T-t)}
\]

*risk-free ZC value*
In the Merton model the **default** occurs when the **firm’s** (assets) value falls below the nominal of its **debt**. The Loss Given Default (LGD) is the expected value of the firm after the debtors are paid.
Merton model and beyond

Pros and cons

Pros:
- Economic interpretation.

Cons:
- There is no conclusion on the optimal amount of the debt;
- The model is very bad for short term default probability;
- Debt structure is too simplistic;
- Debt evolution is exogenous.
Merton model and beyond

Merton model extension

- With jumps;
- With a barrier option approach [Black et al., 1976];
- With a stochastic interest rate [Longstaff et al., 1995];
- With a barrier option approach and stochastic interest rate [Brys et al., 1997];
- Taking into account imperfect information of bond investors [Duffie et al., 1997];
- With an endogenous debt (see next slides).
3 - Leland Model, 1994
Leland Model – Framework

What are the main improvements to Merton model?

Framework of Leland’s model

- At $t = 0$, the owners of a debt-free firm decide to issue debt to optimize their equity value.
- There are two control parameters:
  - $K$ the default trigger;
  - $D_0$ the size of the debt.
- There are two other parameters:
  - $\tau \in [0, 1]$ the tax benefit gained on debt coupons;
  - $\alpha \in [0, 1]$ the fraction of asset value lost at the time of bankruptcy due to frictions.
Leland Model – Framework

What are the main assumptions?

**Leland Model**

- The firm asset value $A_t$ follows a **Geometric Brownian Motion**:

  $$
  \frac{dA_t}{A_t} = (r - \delta) \, dt + \sigma dW_t^Q
  $$

  where $r$ denotes the risk-free rate and $\delta$ the dividend rate

- The debt is a perpetual bond that pays a **constant coupon rate** $C$ every unit of time

- As specified in the contracts (**covenants**), the default of the firm is triggered when $A_t \leq K$:

  $$
  \tau_B = \inf\{t | A_t \leq K\}
  $$

- Prior to the default we always have $E_t \geq 0$, where $E_t$ denotes the value of the equity at time $t$

- At $t = \tau_B$, the debt value $D_{\tau_B}$ is equal to $(1 - \alpha)K$ with $\alpha \in [0, 1[$

- There is a **tax rebate rate** $\tau \in [0, 1]$ on the debt coupons

At $t = 0$, the value of the debt-free company is $A_0$, but the owners have to **surrender a part of their equity** to collect $D_0 > 0$ so that $E_0 < A_0$. The problem for the owners is then to **maximize the firm value** $v_0 = E_0 + D_0 \geq A_0$. 

Loïc BRIN and François CRENIN

Credit Risk - Lecture 3
Merton model – Main results

Model simplifications

Calculation simplifications

As a matter of simplicity:

- We assume that \( Q(\tau_B = \infty) = 0 \), but the conclusions would remain the same without this assumption.
- The value of the firm, its debt and its equity will be computed for \( t = 0 \) to simplify notations.
Merton model – Main results

The value of the debt

Debt value at t=0

Using the risk neutral probability the value of the debt issued at $t = 0$ is equal to the expected discounted cash flows from this debt:

$$D_0 = E^Q \left[ e^{-r \tau_B} (1 - \alpha) K \mathbb{1}_{\{0 \leq \tau_B < \infty\}} \right] + E^Q \left[ \int_0^{\tau_B} C e^{-rt} \, dt \right]$$

(1)

The debt value at $t = 0$ is equal to the expected discounted cash flows from the liquidation value of the assets at $t = \tau_B$ and from the coupons.
Merton model – Main results
The value of the firm

The firm value at t=0
After the issuance of the debt, the firm value is:

\[ v_0 = A_0 + \mathbb{E}^Q \left[ \int_0^{\tau_B} \tau C e^{-rt} dt \right] - \mathbb{E}^Q \left[ e^{-r\tau_B} \alpha K \mathbb{1}_{\{0 \leq \tau_B < \infty\}} \right] \]  (2)

After recapitalization the firm value is equal to the expected discounted cash flows from the tax rebate on coupons minus those lost from the friction when the assets are sold plus the initial value of the assets.
Merton model – Main results

The value of the equity

The equity value at t=0

We can deduce the equity value of the firm at \( t = 0 \) using (1) and (2):

\[
E_0 = v_0 - D_0 = \mathbb{E}^Q \left[ \int_0^{\tau_B} (\delta A_t - (1 - \tau) C) e^{-rt} dt \right]
\]  

(3)

The previous calculation being non-trivial, the equity can also be seen as the expected discounted cash flows from the **dividends** minus those from the fraction of the **coupons** when the tax rebate has been taken into account.
Leland model computations

Laplace transform

Laplace transform of the stopping time

The Laplace transform $L(a, b, \mu) = \Phi_{\tau_B}(a)$ for $\tau_B = \inf\{t \mid W_t + \mu t \geq b\}$ is given by:

$$\Phi_{\tau_B} = \mathbb{E}[e^{-a\tau_B}] = e^{b\left(\mu - \sqrt{\mu^2 + 2a}\right)}$$
Leland model computations

Computation of the Laplace transform of Leland’s bankruptcy time

Since $A_t$ is a **Geometric Brownian Motion** we have:

$$A_t = \exp \left( \ln(A_0) + \left( r - \delta - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right)$$

so that:

$$\mathbb{E}^Q \left[ e^{-r\tau_B} \mathbf{1}_{\{0 \leq \tau_B < \infty\}} \right] = L(r, d_0, -m) = \left( \frac{A_0}{K} \right)^{-\gamma}$$ (4)

where:

$$d_0 = \frac{1}{\sigma} \ln \left( \frac{A_0}{K} \right),$$

$$m = \frac{1}{\sigma} \left( r - \delta - \frac{1}{2} \sigma^2 \right) \leq 0,$$

$$\gamma = \frac{1}{\sigma} \left( m + \sqrt{m^2 + 2r} \right) > 0$$
Leland model computations

Computation of the debt value

From (4) we derive that:

$$E^Q \left[ e^{-r\tau_B} (1 - \alpha) K 1_{\{0 \leq \tau_B < \infty\}} \right] = (1 - \alpha) KL (r, d_0, -m)$$

$$= (1 - \alpha) K \left( \frac{A_0}{K} \right)^{-\gamma}$$

(5)

and

$$E^Q \left[ \int_0^{\tau_B} Ce^{-rt} dt \right] = E^Q \left[ \frac{C}{r} (1 - e^{-r\tau_B}) 1_{\{0 \leq \tau_B < \infty\}} + \frac{C}{r} 1_{\{\tau_B = \infty\}} \right]$$

$$= \frac{C}{r} \left[ 1 - \left( \frac{A_0}{K} \right)^{-\gamma} \right]$$

(6)

From (5) and (6) we have:

Debt value at t=0

$$D_0 = (1 - \alpha) K \left( \frac{A_0}{K} \right)^{-\gamma} + \frac{C}{r} \left[ 1 - \left( \frac{A_0}{K} \right)^{-\gamma} \right]$$

(7)
Leland model computations
Computation of the firm and equity values

In the fashion as for the debt value, we derive from (4) that:

**Firm value at t=0**

\[ v_0 = A_0 + \frac{C \tau}{r} \left[ 1 - \left( \frac{A_0}{K} \right)^{-\gamma} \right] - \alpha K \left( \frac{A_0}{K} \right)^{-\gamma} \]  

(8)

and from (7) and (8) we can conclude that:

**Equity value at t=0**

\[ E_0 = v_0 - D_0 = A_0 - \frac{(1 - \tau) C}{r} \left[ 1 - \left( \frac{A_0}{K} \right)^{-\gamma} \right] - K \left( \frac{A_0}{K} \right)^{-\gamma} \]  

(9)
Prerequisites

- Merton Model, 1973
- Leland Model, 1994

Optimal debt and default trigger

**Optimal debt and default trigger**

Are there optimal debt and default trigger for the shareholders?

By maximizing (9) we get the **optimal** $C$ and $K$ for the **shareholders**:

**How should the owners choose $C$ and $K$?**

\[
K^* (C) = \frac{\gamma(1 - \tau) C}{(\gamma + 1) r} \\
C^* = A_0 \left( \frac{(\gamma + 1) r}{\gamma (1 - \tau)} \left[ \frac{(1 + \gamma) \tau + \alpha (1 - \tau) \gamma}{\tau} \right]^{-\gamma} \right)
\]  

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