Tutorial 2

Structural models

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Exercise 1: Term structure of spreads in Merton's model.

1. Write the value of the debt of the firm for the debt-holders, of the shares of the firm for the shareholders, as an option on the value of the firm and with maturity the maturity of the debt.

We have the following pay offs:

Value of Assets (V_T)	Shareholder's flow	Debt holders's flow
$V_T \geq D$	$V_T - D$	D
$V_T < D$	0	V_T

2. Prove the Call-Put parity in that case.

We know that the put-call parity states that:

$$C_t - P_t = A_t - De^{-r(T-t)}$$

In our case, it means that:

$$\text{Assets} = \text{Debt} + \text{Equity}$$

which is obvious.

3. Give the formula of the price of the shares and the one of the debt of the firm.

By using Black and Scholes results, we have that: Let D be the amount of debt in the balance sheet in t, D_t its value, and E_t the value of equity (in t). We have:

$$D_t = De^{-r(T-t)}\mathcal{N}(d_2) + V_t\mathcal{N}(-d_1)$$

$$E_t = V_t\mathcal{N}(d_1) - De^{-r(T-t)}\mathcal{N}(d_2)$$

with:

$$d_1 = \frac{\log\frac{V_t}{D} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

4. Compute the sensitivity of the price of the share with all the parameters of the problem.

$$\delta = \frac{\partial E_t}{\partial V_t}$$

$$\delta_{Equity} = N(d_1)$$

$$\gamma_{Equity} = \frac{\partial^2 E_t}{\partial V_t^2} = \frac{N'(d_1)}{V_t \sigma \sqrt{T-t}}$$

$$\theta = -\frac{\partial E_t}{\partial T}$$

$$\theta_{Equity} = -\frac{V_t N'(d_1)\sigma}{2\sqrt{T-t}} - rDe^{-r(T-t)}N(d_2)$$

$$\rho = \frac{\partial E_t}{\partial r}$$

$$\rho_{Equity} = D(T-t)e^{-r(T-t)}N(d_2)$$

$$\mathcal{V}_{Equity} = \frac{\partial E_t}{\partial \sigma} = V_t \sqrt{T-t} N'(d_1)$$

5. Compute the term structure of spreads in Merton's model and comment the results.

We have that:

$$D_t = De^{-(r+s_t)(T-t)}$$

Thus:

$$s_t = \frac{1}{T-t} \ln\left(\frac{D}{D_t}\right) - r$$

with D_t expressed earlier.

We have that when σ grows, D_t decreases and thus the spread s_t increases.

Exercise 2: Leland's model (1994).

In the following exercise, the underlying model is Leland's, in its easiest form (1994).

Let us consider a firm which asset value follows, under the historical probability, the following diffusion:

$$\frac{\mathrm{d}A_t}{A_t} = (\mu - \delta)\mathrm{d}t + \sigma \mathrm{d}W_t$$

where W_t is a standard brownian motion.

The firm is financed through debt and equity that are exchanged on financial markets. Debt has a simple structure: the nominal is D, it was borrowed at $t = 0$ until the end of time, and the firm has to pay a coupon, C, until the end of time.

The firm is managed by the shareholders. These can, at any moment, decide to stop the activity and trigger the bankruptcy.

r is the risk free rate.

1. In the case the firm will never default, what is the value of the debt?

In the case the firm will never default, we have that the value of the debt is: $\int_0^{+\infty} C \times e^{-rt}\mathrm{d}t = \frac{C}{r}$.

2. We recall that the Laplace transform $L(a, b, \mu) = \Phi_{\tau_B}(a)$ for $\tau_B = \inf\{t \mid W_t + \mu t \geq b\}$ is given by:

$$\Phi_{\tau_B} = \mathbb{E}\left[e^{-a\tau_b}\right] = e^{b\left(\mu - \sqrt{\mu^2+2a}\right)}$$

Since A_t is a Geometric Brownian Motion we recall that:

$$A_t = \exp\left(\ln(A_0) + \left(r - \delta - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$$

Compute the value for D_0.

The debt value at $t = 0$ is equal to the expected discounted cash flows from the liquidation value of the assets at $t = \tau_B$ and from the coupons. Thus:

$$D_0 = \mathbb{E}^Q\left[e^{-r\tau_B}(1-\alpha)K\mathbb{1}_{\{0\leq\tau_B<\infty\}}\right] + \mathbb{E}^Q\left[\int_0^{\tau_B} Ce^{-rt}dt\right]$$

So we have that:

$$\mathbb{E}^Q\left[e^{-r\tau_B}\mathbb{1}_{\{0\leq\tau_B<\infty\}}\right] = L(r, d_0, -m) = \left(\frac{A_0}{K}\right)^{-\gamma} \tag{1}$$

where:

$$\begin{aligned} d_0 &= \frac{1}{\sigma}\ln\left(\frac{A_0}{K}\right), \\ m &= \frac{1}{\sigma}\left(r - \delta - \frac{1}{2}\sigma^2\right) \leq 0, \\ \gamma &= \frac{1}{\sigma}\left(m + \sqrt{m^2 + 2r}\right) > 0 \end{aligned}$$

We derive that:

$$\begin{aligned} \mathbb{E}^Q\left[e^{-r\tau_B}(1-\alpha)K\mathbb{1}_{\{0\leq\tau_B<\infty\}}\right] &= (1-\alpha)KL(r, d_0, -m) \\ &= (1-\alpha)K\left(\frac{A_0}{K}\right)^{-\gamma} \end{aligned} \tag{2}$$

and

$$\begin{aligned} \mathbb{E}^Q\left[\int_0^{\tau_B} Ce^{-rt}dt\right] &= \mathbb{E}^Q\left[\frac{C}{r}\left(1-e^{-r\tau_B}\right)\mathbb{1}_{\{0\leq\tau_B<\infty\}} + \frac{C}{r}\mathbb{1}_{\{\tau_B=\infty\}}\right] \\ &= \frac{C}{r}\left[1 - \left(\frac{A_0}{K}\right)^{-\gamma}\right] \end{aligned} \tag{3}$$

We thus have:

$$D_0 = (1-\alpha)K\left(\frac{A_0}{K}\right)^{-\gamma} + \frac{C}{r}\left[1 - \left(\frac{A_0}{K}\right)^{-\gamma}\right] \tag{4}$$

Paying the debt induces a fiscal rebate of τC for each period of time, where τ is the abatement rate. Furthermore, a fraction α of the assets is lost as bankruptcy costs in case of failure of the firm.

3. Compute the market value of the fiscal abatement, and its bankruptcy costs.

The no-arbitrage hypothesis on (4) let us identify $\left(\frac{A_0}{K}\right)^{-X}$ as the probability that bankruptcy happens. The market value of the fiscal abatement is then:

$$\frac{\tau C}{r}\left(1 - \left(\frac{A_0}{K}\right)^{-X}\right)$$

And the market value of the bankruptcy costs is:

$$\alpha K\left(\frac{A_0}{K}\right)^{-X}$$

4. What is the value of a zero-coupon that pays 1 EUR in case of failure of the firm.

The value of a zero-coupon bond is:

$$\left(\frac{V_0}{V_B}\right)^{-X}$$

Adding to the assets, the value of the fiscal abatement and deducting the costs of bankruptcy, we get the total market firm value.

5. Give the total market value of the firm.

The total value of the firm at $t = 0$ is:

$$v = V_0 + \frac{\tau C}{r}\left(1 - \left(\frac{V_0}{K}\right)^{-X}\right) - \alpha K\left(\frac{V_0}{K}\right)^{-X}$$

6. Debt was sold at par, that is $P = \frac{C}{r}$. Compute the market value of the firm.

As debt was sold at par, we have that:

$$D(V_0) = \frac{C}{r}\left(1 - \left(\frac{V_0}{K}\right)^{-X}\right) + (1-\alpha)K\left(\frac{V_0}{K}\right)^{-X}$$

7. Compute the market value of the shares.

Thus, the equity value is:

$$E(V_0, K) = v(V_0, K) - D(V_0, K) = V_0 - (1-\tau)\frac{C}{r} + \left((1-\tau)\frac{C}{r} - K\right)(V_0K)^{-X}$$

Shareholders solve the following program:

$$K = \text{argmax}_{V_B}(E(V_0, K))$$

and:

$$K = (1-\tau)\frac{c}{r(X+1)}$$