Tutorial 4

Portfolio Models and ABS

École Nationale des Ponts et Chausées
Département Ingénieurie Mathématique et Informatique – Master II

Loïc BRIN • Benoit ROGER

Exercise 1: From granular homogeneous portfolio to non-homogeneous portfolio.

First, we consider a granular homogeneous pool, with internal risk parameters : PD $= \mathcal{N}(s)$, LGD and ρ for the correlation.

1. Compute the mean and the standard deviation of losses on the portfolio (Hint: consider a finite number, N, of assets first and then generalize).

> Let us consider N assets, with the same nominal $1/N$ and maturity, which default occur when $D_i = \mathbb{1}_{R_i<s}$ where s is the common threshold and R_i a normal variable equal to: $R_i = \sqrt{\rho}F + \sqrt{1-\rho}\epsilon_i$, where F and ϵ_i are independent.
>
> Let L_N be the loss of this portfolio. We have:
>
> $$L_N = \frac{\text{LGD}}{N} \times \sum_{i=1}^{N} \mathbb{1}_{R_i \leq s}$$
>
> We know that R_i are standard normal variables:
>
> $$\begin{aligned} \mathbb{E}(L_N) &= \frac{LGD}{N} \sum_{i=1}^{N} \mathbb{E}(\mathbb{1}_{R_i<s}) \\ &= \frac{LGD}{N} \sum_{i=1}^{N} \underbrace{\mathbb{P}(R_i < s)}_{\Phi(s)} \\ &= LGD \times \Phi(s) \end{aligned}$$
>
> By taking the limit we deduce that:
>
> $$\begin{aligned} \mathbb{E}(L) &= \lim_{N\to\infty} LGD \times \Phi(s) \\ &= LGD \times \Phi(s) \end{aligned}$$

$\mathbb{E}(L^2)$ is obtained as follow:

$$\begin{aligned}
\mathbb{E}({L_N}^2) &= \frac{LGD^2}{N^2}\mathbb{E}\left[\left(\sum_{i=1}^{N}\mathbb{1}_{R_i<s}\right)^2\right] \\
&= \frac{LGD^2}{N^2}\mathbb{E}\left[\sum_{i=1}^{N}\mathbb{1}_{R_i<s} + 2\sum_{0\le i<j\le N}\mathbb{1}_{R_i<s}\mathbb{1}_{R_j<s}\right] \\
&= \frac{LGD^2}{N^2}\left[\sum_{i=1}^{N}\mathbb{E}(\mathbb{1}_{R_i<s}) + 2\sum_{0\le i<j\le N}\mathbb{E}\left(\mathbb{1}_{R_i<s}\mathbb{1}_{R_j<s}\right)\right] \\
&= \frac{LGD^2}{N^2}\left[\sum_{i=1}^{N}\underbrace{\mathbb{P}(R_i<s)}_{\Phi(s)} + 2\sum_{0\le i<j\le N}\underbrace{\mathbb{P}\left(R_i<s, R_j<s\right)}_{\Phi_2(s,s,\rho)}\right] \\
&= LGD^2 \times \left[\frac{\Phi(s)}{N} + \frac{N(N-1)}{N^2}\Phi_2(s,s,\rho)\right]
\end{aligned}$$

By taking the limit we deduce that:

$$\begin{aligned}
\mathbb{E}(L^2) &= \lim_{N\to\infty} LGD^2 \times \left[\underbrace{\frac{\Phi(s)}{N}}_{\to 0} + \underbrace{\frac{N(N-1)}{N^2}}_{\to 1}\Phi_2(s,s,\rho)\right] \\
&= LGD^2 \times \Phi_2(s,s,\rho)
\end{aligned}$$

Hence, we deduce the variance of the portfolio loss:

$$\begin{aligned}
\mathbb{V}(L) &= \mathbb{E}(L^2) - \mathbb{E}(L)^2 \\
&= LGD^2 \times \underbrace{\left(\Phi_2(s,s,\rho) - \Phi(s)^2\right)}_{\ge 0,\text{ since } \rho\ge 0}
\end{aligned} \tag{1}$$

2. For each $\alpha \in [0;1]$, compute the α loss quantile.

We need to find x such as:

$$\mathbb{P}(L \le x) = \alpha$$

<u>First method:</u>

We have seen that:

$$L = LGD \cdot \Phi\left(\frac{s-\sqrt{\rho}F}{\sqrt{1-\rho}}\right) \tag{2}$$

Thus:

$$\begin{aligned}
\mathbb{P}(L \le x) &= \mathbb{P}\left(LGD \cdot \Phi\left(\frac{s-\sqrt{\rho}F}{\sqrt{1-\rho}}\right) \le x\right) \\
&= \mathbb{P}\left(F \ge \frac{s-\sqrt{1-\rho}\,\Phi^{-1}(\frac{x}{LGD})}{\rho}\right) \qquad (3) \\
&= \underbrace{\Phi\left(\frac{\sqrt{1-\rho}\,\Phi^{-1}\left(\frac{x}{LGD}\right)-s}{\sqrt{\rho}}\right)}_{\text{portfolio loss cdf}} \qquad (4)
\end{aligned}$$

(3) to (4) comes from the fact that F is a symmetrical variable $(1-\Phi(x)=\Phi(-x))$.

Therefore:

$$\begin{aligned}
&\mathbb{P}(L \leq x) = \alpha \\
\iff & \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}\left(\frac{x}{LGD}\right)-s}{\sqrt{\rho}}\right) = \alpha \\
\iff & x = LGD \cdot \Phi\left(\frac{s+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)
\end{aligned}$$

We then have:

$$q_L(\alpha) = LGD \cdot \Phi\left(\frac{s+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)$$

<u>Second method:</u>

We can notice that L is decreasing with F which is the only stochastic term in (2) and since the quantile function of F is Φ^{-1},therefore:

$$q_L(\alpha) = LGD \cdot \Phi\left(\frac{s-\sqrt{\rho}\Phi^{-1}(1-\alpha)}{\sqrt{1-\rho}}\right)$$

F is symmetrical therefore $\Phi^{-1}(1-\alpha) = -\Phi^{-1}(\alpha)$ and:

$$q_L(\alpha) = LGD \cdot \Phi\left(\frac{s+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)$$

3. For a given threshold $l_0 <$ LGD, compute $\mathbb{E}[(L-l_0)^+]$.

$$\begin{aligned}
\mathbb{E}\left[(L-l_0)^+\right] &= \mathbb{E}\left[(L-l_0)\cdot \mathbb{1}_{L>l_0}\right] \\
&= \mathbb{E}\left[L\cdot \mathbb{1}_{L>l_0}\right] - l_0 \cdot \underbrace{\mathbb{E}\left[\mathbb{1}_{L>l_0}\right]}_{=\mathbb{P}(L>l_0)}
\end{aligned} \tag{5}$$

We have:

$$\begin{aligned}
\mathbb{E}\left[L\cdot \mathbb{1}_{L>l_0}\right] &= \mathbb{E}\left[L\cdot \mathbb{1}_{LGD\cdot\Phi\left(\frac{s-\sqrt{\rho}F}{\sqrt{1-\rho}}\right)\geq l_0}\right] \\
&= \mathbb{E}\left[L\cdot \mathbb{1}_{F\leq \frac{s-\sqrt{1-\rho}\Phi^{-1}\left(\frac{l_0}{LGD}\right)}{\sqrt{\rho}}}\right] \\
&= \mathbb{E}\left[\lim_{N\to\infty}\frac{LGD}{N}\sum_{i=1}^{N}\mathbb{1}_{R_i<s}\cdot \mathbb{1}_{F\leq \frac{s-\sqrt{1-\rho}\Phi^{-1}\left(\frac{l_0}{LGD}\right)}{\sqrt{\rho}}}\right] \\
&= \lim_{N\to\infty}\frac{LGD}{N}\sum_{i=1}^{N}\mathbb{E}\left[\mathbb{1}_{R_i<s}\cdot \mathbb{1}_{F\leq \frac{s-\sqrt{1-\rho}\Phi^{-1}\left(\frac{l_0}{LGD}\right)}{\sqrt{\rho}}}\right] \\
&= \lim_{N\to\infty}\frac{LGD}{N}N\Phi\left(s, \frac{s-\sqrt{1-\rho}\Phi^{-1}\left(\frac{l_0}{LGD}\right)}{\sqrt{\rho}}, \rho\right) \\
&= LGD\cdot\Phi\left(s, \frac{s-\sqrt{1-\rho}\Phi^{-1}\left(\frac{l_0}{LGD}\right)}{\sqrt{\rho}}, \rho\right)
\end{aligned} \tag{6}$$

From (5) and (7) we can conclude:

$$\mathbb{E}\left[(L-l_0)^+\right] = LGD\left[\cdot\Phi\left(s, \frac{s-\sqrt{1-\rho}\Phi^{-1}\left(\frac{l_0}{LGD}\right)}{\sqrt{\rho}}, \rho\right) - l_0\cdot\Phi\left(\frac{s-\sqrt{1-\rho}\Phi^{-1}\left(\frac{l_0}{LGD}\right)}{\sqrt{\rho}}\right)\right] \quad (7)$$

4. We now relax the homogeneity assumption and consider that each asset has its own default probability asset i defaults as soon as its asset return $R_i = \sqrt{\rho}F + \sqrt{1-\rho}\epsilon_i < s + \sigma\bar{\epsilon}_i$, where all random variables $(F, (\epsilon_i)_i, (\bar{\epsilon}_i)_i)$ are normal, centered, reduced, independent variables.
Show that the loss distribution remains a Vasicek loss distribution, with modified parameters.

$$s' = \frac{s}{\sqrt{1+\sigma^2}}$$

$$\rho' = \frac{\rho}{1+\sigma^2}$$

Comment.

$$\begin{aligned}
R_i < s + \sigma\bar{\epsilon}_i \quad &\iff \quad \sqrt{\rho}F + \sqrt{1-\rho}\epsilon_i < s + \sigma\bar{\epsilon}_i \\
&\iff \quad \sqrt{1-\rho}e_i - \sigma\bar{\epsilon}_i < s - \sqrt{\rho}F \\
&\iff \quad \frac{\sqrt{1-\rho}e_i - \sigma\bar{\epsilon}_i}{\sqrt{1-\rho+\sigma^2}} < \frac{s-\sqrt{\rho}F}{\sqrt{1-\rho+\sigma^2}} \\
&\iff \quad \frac{\sqrt{1-\rho}e_i - \sigma\bar{\epsilon}_i}{\sqrt{1-\rho+\sigma^2}} < \frac{s-\sqrt{\rho}F}{\sqrt{1+\sigma^2}\sqrt{1-\frac{\rho}{1+\sigma^2}}} \\
&\iff \quad \frac{\sqrt{1-\rho}e_i - \sigma\bar{\epsilon}_i}{\sqrt{1-\rho+\sigma^2}} < \frac{\frac{s}{\sqrt{1+\sigma^2}} - \sqrt{\frac{\rho}{1+\sigma^2}}F}{\sqrt{1-\frac{\rho}{1+\sigma^2}}}
\end{aligned}$$

$\frac{\sqrt{1-\rho}e_i - \sigma\bar{\epsilon}_i}{\sqrt{1-\rho+\sigma^2}}$ is Gaussian as the sum of two Gaussian variables. Moreover, it centered with a variance equal to 1. It is therefore a standard Gaussian variable. We can then conclude.

We can see that $\rho' = \frac{\rho}{1+\sigma^2} < \rho$. Hence, adding a noise increases the idiosyncratic risk compared to the systemic risk. Noticing that $\Phi\left(s', s', \rho'\right) - \Phi(s')^2 \leq \Phi\left(s, s, \rho\right) - \Phi(s)^2$ we can deduce from (1) that adding heterogeneity to the thresholds (or PDs) within the portfolio decreases the variance of the portfolio loss. The diversification is therefore increased.

Exercise 2: Analysis of a securitization deal.

You are working in a bank and want to compute the required regulatory capital of a securitization deal. The deal consists in the securitization of a portfolio of mortgage loans which have the same size and the same maturity (7 years).

In this tutorial, we are going to use a simplified version of rating agencies methodologies. It consists in estimating for each loan a probability of default and an hypothesis in case of major changes in the market. These depend on the initial amount provided (1 - Loan To Value), the ratio Debt-to-Income, the geographical zone and data to assess a trend and volatility on the real estate market.

The tranching of the deal is as follows:
We will use the following parameters:

- Probability of default for each borrower : 2 %;

Class	Amount	Owner
A	900	Investors
B	70	Investors
C	30	Bank
TOTAL	**1000**	

- Loan to value: 75 %;
- Market-Value-Decline: 30 %;
- Repossession costs: 10 %.

1. What is the SFA capital on the pool before securitization?

In order to compute the SFA capital, we need several parameters:

- the Probability of Default (PD): 2 %;
- the Loss Given Default (LGD): for a 100 mortgage loan, 30 would be lost for market value decline, 10 for repossession costs, so the recovered amount would be 60 for an exposure of 75 (Loan To Value of 75 %). That means that the LGD is $\frac{15}{75} = 20\%$.

The SFA formula for mortgage exposure is:

$$R = 0.15$$

$$K = \text{LGD} * \left[N\left(\sqrt{\frac{1}{1-R}} * G(\text{PD}) + \sqrt{\frac{R}{1-R}} * G(0.999) \right) - \text{PD} \right]$$

where:

- R is a correlation parameter;
- K is the capital;
- $N(x)$ denotes the normal cumulative distribution function;
- $G(z)$ denotes the inverse cumulative distribution function.

2. A rating is given by agency ratings under the following conditions:

Number of time the expected loss is covered	Rating
1	BB
2	BBB
3	A
4	AA
5 or more	AAA

What are the ratings of the different tranches? What can you say about the methodology?

We first need to compute the one-year expected loss:

$$\mathbb{E}(L_{1\text{ year}}) = \text{EAD} \times \text{LGD} \times \text{PD} = 1000 \times 20\% \times 2\% = 4$$

We then have to determine the Total Expected Loss (i.e. on the full life of transaction).

As we have no other information on the amortization profile, let us consider that:

$$\text{Total Loss} = \mathbb{E}(L_{1\text{ year}}) \times 7 \text{ years} = 28$$

This is a conservation assumption.

A BBB tranche should have $2 \times$ Total Loss as credit enhancement. That is, a BBB tranche should have a $28 \times 2 = 56$ Credit Enhancement.

Hence, we deduce that the tranches are rated as follows:

- C: Non rated;
- B: covers 1 Expected Loss and is therefore BB rated;
- A: has a credit enhancement equal to nearly 4 times Total Expected Loss. It is then rated AA.

In practice, because of amortization profile, Average Maturity will be less than 7 years, that would lead to a better rating than the ones that have been computed here above.

3. Propose another model to rate the tranches, assuming that you know the probabilities of default of all the corporate ratings.

What we could do, is to make an hypothesis on the correlation of the defaults of mortgages in the pool and use a Vasicek model (as we already know PD and LGD). On that basis, we would compute an expected loss (as a percentage) of each tranche that we could compare to the probabilities of all the corporate ratings in order to determine a grade.

4. Same question but assuming that there is an excess spread of 50 bp on the pool.

Reminder about excess spread: its corresponds to a part of the interests that are not paid to tranches holders but set aside so that in case of default, the amount can cover part of the losses. It is a technique to enhance credit.

The problem gets trickier as we have to simulate the default of the pool on a one-year horizon in order to take into account the flow of the excess spread and its consequent effect on the losses of tranche C, B and A.

On that basis, we would compute a one-year default probability that would take this credit enhancement into account and compare it to the corporate ratings.

5. Knowing that the loans can be reimburse in advance, what is the effect on senior tranches?

The effect on the quality on senior tranches is positive. Indeed, would half the portfolio be reimbursed, the structure of the portfolio would be the following:

Class	**Amount**	**Owner**
A	400	Investors
B	70	Investors
C	30	Bank
TOTAL	**500**	

That means that the credit enhancement generated by tranching the asset pool is ore protective for the senior tranche, in proportion, as tranche A and B represent $\frac{100}{500} = 20\%$ instead of $\frac{100}{1000} = 10\%$.